Monte Carlo study of correlated continuum percolation: Universality and percolation thresholds

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Critical exponents of the continuum-percolation system of two-dimensional distributions of disks in the penetrable concentric shell model are determined by Monte Carlo simulations and by real-space Monte Carlo renormalization-group techniques. In this model, each disk of diameter σ is composed of a mutually impenetrable core of diameter $\lambda\sigma$ ($0\le\lambda\le1$) encompassed by a perfectly penetrable shell of thickness $(1-\lambda)\sigma/2$. Pairs of particles are connected when the interparticle distance is less than σ. We find that the susceptibility exponent γ is given by 2.50 ± 0.03 for an impenetrability parameter $\lambda=0.8$ and the correlation-length exponent ν to be between 1.30 and 1.35 for various values of λ. Both results consistently suggest that continuum percolation in the penetrable concentric shell model for nonzero hard-core radii belongs to the same universality class as that of ordinary lattice percolation and of randomly centered disks, as far as the geometrical critical exponents are concerned. We also present the critical reduced number densities and critical area fractions for selected values of λ.

I. INTRODUCTION

The preponderance of percolation investigations have treated percolation on lattices[1,2] in which each site or bond is occupied with probability p and unoccupied with probability $1-p$ (see Refs. 1 and 2 and references therein). More recently, investigators have focused their attention on continuum models of percolation since such models, in many cases, are better able to capture the essential physics of real systems.[3-11] It was relatively recently that the question regarding the universality of continuum and lattice percolation was addressed.[4,5] In the prototypical continuum-percolation model, randomly centered (or spatially uncorrelated) particles are distributed in space and a bond is assumed to exist between two such particles if they overlap. Monte Carlo (MC) simulations[4] and *real-space* Monte Carlo renormalization-group[5] (MCRG) approaches were employed to estimate various critical exponents and percolation thresholds for continuum percolation of such *freely* overlapping disks. All geometrical critical exponents were found to be similar to those of lattice systems, dispelling possible concerns that freely overlapping particles and lattice percolation might be in different universality classes.

A natural question which arises is the following: What are the effects of exclusion-volume interactions on the critical exponents of continuum-percolation systems? For two-dimensional distributions of squares in the penetrable concentric shell (PCS) model,[12] this question was examined by Gawlinski and Redner.[4] However, as Gawlinski and Redner noted, the accuracy of their estimates were not sufficiently precise to determine whether there is universality of continuum systems with respect to exclusion-volume effects.

One of the purposes of this paper is to determine, with high accuracy, the effect of exclusion-volume interactions on the critical exponents for two-dimensional continuum percolation of distributions of *circular disks* at number density ρ in the PCS model.[12] In the PCS model, each disk of diameter σ is composed of a mutually impenetrable core of diameter $\lambda\sigma$, encompassed by a perfectly penetrable shell of thickness $(1-\lambda)\sigma/2$, with $0\le\lambda\le1$ (see Fig. 1). The extreme limits of $\lambda=0$ and $\lambda=1$ correspond, respectively, to the cases of fully penetrable (i.e., spatially uncorrelated or randomly centered particles) and totally impenetrable particles. Thus, by varying λ continuously between zero and unity, one can vary the degree of impenetrability (or exclusion-volume effects) and, hence, the degree of the connectedness of the particles. The critical exponents, such as the susceptibility and correlation-length exponents, may or may not depend upon the impenetrability parameter λ.

For λ near but not equal to zero, one might expect such properties not to be very different from those of fully penetrable particles ($\lambda=0$). However, as λ is made larger and approaches unity, or in the extreme case of $\lambda=1$, the mean cluster size and percolation susceptibility increase very rapidly as the inclusion area fraction ϕ approaches the critical area fraction ϕ_c, suggesting a possibly large increase in the susceptibility exponent γ. (Note that for an equilibrium distribution of totally impenetrable particles, the mean cluster size is precisely unity

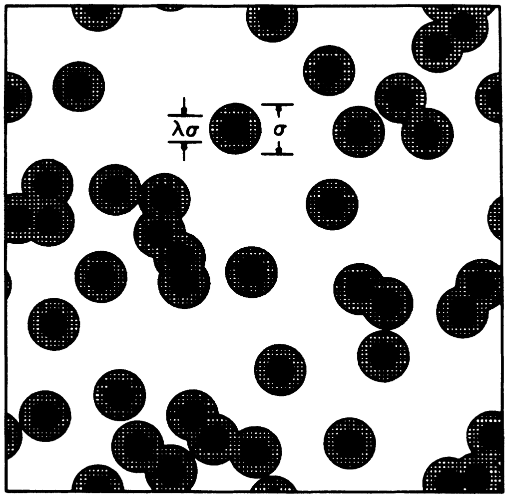

FIG. 1. A computer-generated realization of a distribution of disks of radius $\sigma/2$ (shaded region) in the PCS model. The disks have a hard core of diameter $\lambda\sigma$ indicated by the smaller, black circular region. Here $\lambda=0.5$ and the area fraction of disks is approximately 0.3.

for $\phi<\phi_c$; however, at $\phi=\phi_c$, i.e., the random close-packing fraction, it diverges suddenly.) If this behavior is indeed the case, one might question where the "unstable" fixed point is located. It is at this "critical" point, if any, that the critical properties of continuum percolation for the PCS model are expected to be different from those of the fully penetrable case.

We specifically determine by MC simulations and MCRG transformations the susceptibility exponent γ and correlation-length exponent ν defined, respectively, by

$$\chi \sim |\phi-\phi_c|^{-\gamma} \tag{1}$$

and

$$\xi \sim |\phi-\phi_c|^{-\nu} , \tag{2}$$

for distributions of disks with impenetrability parameter λ and number density ρ. Note that one can replace the area fractions ϕ and ϕ_c with the reduced number densities η and η_c, respectively, in Eqs. (1) and (2) (where $\eta=\rho\pi\sigma^2/4$) since ϕ is an analytic function of η for $0\le\lambda\le 1$.[13] MC simulations are carried out to estimate the exponent γ below and above ϕ_c or η_c for a particular choice of λ ($\lambda=0.8$) and system size L ($L=50$). MCRG transformations are performed to calculate the exponent ν for various values of λ. Based on our results for these two critical exponents, we conclude that the PCS model is in the same universality class as ordinary lattice percolation. We also estimate the values of the percolation threshold ϕ_c or η_c for selected values of the impenetrability parameter λ.

In Sec. II, we describe the MCRG method employed to obtain ν and η_c for selected values of λ between 0 and 1. In Sec. III, we describe the MC procedure used to generate equilibrium continuum realizations. In Sec. IV, we present the results of our MC simulations and MCRG transformations. We give estimates of the critical exponents ν and γ and the percolation thresholds η_c extrapolated to the infinite-system limit. The critical area fractions ϕ_c are also estimated from η_c for various values of λ. In Sec. V, we summarize our major findings and make concluding remarks.

II. MONTE CARLO RENORMALIZATION-GROUP METHOD

Consider a distribution of equisized disks of unit diameter and an inner impenetrable core of diameter λ. A rescaling is performed in which a $b\times b$ cell is mapped onto a $b'\times b'$ cell with the same core fraction for each disk. The recursion relation of such a one-parameter cell-to-cell transformation can be written as

$$R'(p',b')=R(p,b) , \tag{3}$$

where $R(p,b)$ is the connectivity function, i.e., the probability that there is a spanning cluster at concentration p which connects any two opposite edges across the cell. A parameter p is, in its original definition for lattice percolation, the fraction of occupied sites at which a spanning cluster appears for the first time. In continuum percolation, it is naturally considered to be the overall inclusion area fraction for which a spanning cluster appears for the first time.

In lattice percolation, obtaining $R(p,b)$ is simple. One can take a $b\times b$ cell and fill it on randomly chosen sites until a spanning cluster appears across the cell. The parameter p is identified as the fraction of occupied sites when spanning cluster appears for the first time. Repeating this procedure thousands of times, one can obtain the distribution of p values which approximates the underlying probability density function $L(p,b)$. The connectivity function is obtained as

$$R(p,b)=\int_0^p L(\hat{p},b)d\hat{p} . \tag{4}$$

In continuum percolation, the inclusion area fraction ϕ plays the role of the parameter p in the renormalization recursion relation (3). However, a formulation in terms of ϕ has a drawback in that the area fraction is not a controllable simulation parameter for the PCS model. The area fraction of inclusions for any finite-sized system varies from realization to realization for a given reduced number density η because of different degrees of overlap among the particles. Moreover, in the Metropolis algorithm[14] which we employ here, one cannot increase, for fixed λ, the number density of inclusions continuously during the simulation, unlike the case of lattice percolation.

In order to avoid this difficulty, we choose the reduced number density η as a renormalization parameter and obtain the connectivity function as a function of η. (The quantity η is related to ϕ as discussed in Sec. IV and Ref. 13.) We generate systems of N particles (thus fixing η) in a given cell of size b and sample realizations, following the method to be described in Sec. III. Over thousands of realizations, we search for spanning clusters and count the number of realizations having such clusters. This number gives the approximate probability of having spanning clusters at a given density. Repeating this whole procedure over many different values of η and spline fitting the results, we obtain a continuous connectivity function $R(\eta,b)$.

Once $R(\eta,b)$ is obtained both in $b\times b$ and $b'\times b'$ cells, the renormalization procedure is straightforward. The fixed point $\eta^*_{b,b'}$ of the recursion relation (3) is the intersection of $R(\eta,b)$ and $R(\eta,b')$ and it gives the approximate critical point at which the system begins to percolate. The correlation-length exponent ν can be obtained, for a particular choice of b and b', from

$$\nu=\ln(b/b')/\ln\Lambda_{b,b'} , \tag{5}$$

where

$$\Lambda_{b,b'}=\left[\frac{dR(\eta,b)}{d\eta}\right]\Bigg/\left[\frac{dR(\eta,b')}{d\eta}\right]\Bigg|_{\eta=\eta^*_{b,b'}} \tag{6}$$

is the eigenvalue of the linearized transformation. Following Reynolds, Stanley, and Klein[15] one can obtain extrapolations of ν to the $b/b'\rightarrow 0$ limit where the renormalization should be exact:

$$\ln\Lambda_{b,b'}=y\ln(b/b')+c . \tag{7}$$

Here $y=1/\nu$ and c is a constant. Thus one can determine y from the slope of a plot of $\ln\Lambda_{b,b'}$ versus $\ln(b/b')$. In the same limit, $R(\eta,b)$ approaches a step function and a jump discontinuity occurs at $\eta^*_{b,b'}=\eta_c$. The deviation of $\eta^*_{b,b'}$ from η_c for any finite system is known to scale as

$$\Delta\eta=|\eta^*_{b,b'}-\eta_c|\sim(b/b')^{-1/\nu} . \tag{8}$$

Therefore η_c is also determined from a plot of $(b/b')^{-1/\nu}$ versus $\eta^*_{b,b'}$.

III. MONTE CARLO PROCEDURE

We generate equilibrium realizations of the PCS model employing the well-known Metropolis algorithm.[14] Disks of unit diameter and inner impenetrable core of diameter λ are initially placed, with no hard-core overlaps, on the sites of a triangular lattice in a $b\times b$ square cell. The cell was surrounded by periodic images of itself. Each particle was then moved to some new position which was accepted or rejected according to whether or not the inner hard cores overlapped. This process was repeated many times until equilibrium was achieved. Each of our simulation consists of 200 moves per particle before sampling for equilibrium properties. Equilibrium realizations were sampled at every 20 moves per particle. In order to ensure that equilibrium was achieved, we determined the pressure as a function of η for systems of particles having diameter $\lambda\sigma$ ($\sigma=1$). The pressures obtained were in very good agreement with previous accurate determinations of the pressure.[16]

A. Monte Carlo simulations

In order to estimate susceptibility exponent γ, we generate equilibrium realizations for $\lambda=0.8$ in a given cell of size $L=50$. The number of particles used ranged between 1800 ($\eta=0.565$) and 2740 ($\eta=0.861$). In order to calculate the percolation susceptibility, one must make use of an algorithm which distinguishes between the various clusters in the system. By definition, two particles are assumed to be "directly" connected if their interparticle distance is less than σ ($\sigma=1$). Pairs of particles may be "indirectly" connected, however, i.e., pairs can be connected through chains of other particles. Existing cluster-counting algorithms which can distinguish such clusters include the "cluster-labeling" method[17] and the "connectivity-matrix" method.[10] The former was originally developed for lattice percolation and subsequently adapted for continuum percolation.[4] The latter was developed for continuum percolation but can be applied to lattice percolation as well. In this study, we employ a modified cluster-labeling method. The susceptibility is obtained according to the definition

$$\chi=\sum_s{}' n_s s^2 , \tag{9}$$

where the prime means that the biggest cluster was eliminated for $\eta>\eta_c$ and n_s denotes the mean number of clusters of size s per particle.

B. MCRG transformations

Consider a $b\times b$ cell of N particles. In order to establish MCRG transformations, we generate systems of $b=5$, 6, 7, 10, 15, 20, 25, 30, and 40 for selected values of λ between 0 and 1; $\lambda=0$, 0.3, 0.5, 0.7, 0.8, and 0.9. The number of particles used varied widely from 17 (for $\lambda=0.9$ and $b=5$) to as many as 2350 (for $\lambda=0$ and $b=40$) depending on b and η. The reduced number density η is given as $\eta=N\pi/4b^2$. In each cell, we generate 500–20 000 realizations and calculate what fraction of realizations include spanning clusters across the cell, thus obtaining $R(\eta,b)$ given in Eq. (3).

IV. RESULTS AND DISCUSSION

In order to estimate the susceptibility exponent γ, we have carried out MC simulations for the PCS model for a particular choice of impenetrability parameter $\lambda=0.8$, in a given system of size $L=50$. The percolation susceptibility defined by Eq. (9) was calculated in each realization and the final result was obtained by averaging over 200 realizations for each selected η, given as $\eta=N\pi/4L^2$.

Simulation data are plotted in Fig. 2 on a log-log scale as a function of $|1-\eta/\eta_c|$. The percolation threshold used is $\eta_c=0.7685$, which was estimated in such a way that susceptibilities below and above η_c produce parallel lines in this plot. This value of η_c is considerably greater than the accurate determination of it by the finite-size scaling analysis, $\eta_c=0.7533\pm0.0003$ (see the results and discussion below). The large value of η_c used in this plot is consistent with the general trends that percolation thresholds are significantly overestimated if obtained from the critical behavior of mean cluster size (and correspondingly susceptibility) for any *finite* system.[11] This is

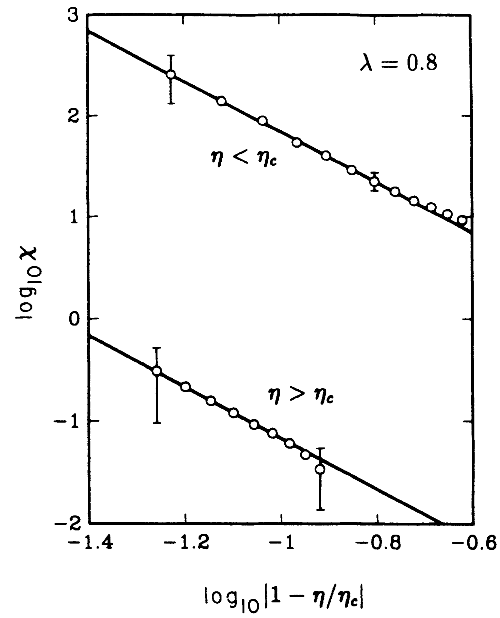

FIG. 2. Double logarithmic plot of the percolation susceptibility χ against $|1-\eta/\eta_c|$ for $\lambda=0.8$ and $L=50$. Parallel straight lines were obtained at $\eta_c=0.7685$ and the critical exponent γ estimated from the slope is $\gamma=2.50\pm0.03$.

expected since MC simulations underestimate the mean cluster size for any finite-sized system, and thus, the long-ranged behavior of the pair-connectedness function begins to occur for η greater than true percolation point.

The critical exponent γ and amplitude ratio R for the susceptibility above and below η_c are estimated from the slope and the vertical displacement of two parallel lines in Fig. 2. We find

$$\gamma=2.50\pm0.03 \tag{10}$$

and

$$R=C_+/C_-=1050\pm32 \ . \tag{11}$$

In both cases, the quoted errors are those associated with linear regression. This value of γ is reasonably close to the lattice value ($\gamma=2.43\pm0.03$, cf. Ref. 18) and is also comparable to the continuum values for fully penetrable disks ($\gamma=2.43\pm0.04$, cf. Ref. 4), random bond percolation of points ($\gamma=2.50$, cf. Ref. 19) and penetrable widthless sticks ($\gamma=2.55\pm0.08$, cf. Ref. 20). On the other hand, the amplitude ratio R, which is supposed to be universal, is significantly different from the continuum value of randomly centered disks ($R=50$, cf. Ref. 4). This situation is quite similar to the cases of lattice-continuum percolation and percolation-gelation analogies.[21] In continuum percolation of randomly centered disks, Gawlinski and Stanley[4] asserted on the basis of MC data that all critical exponents are the same as those of ordinary lattice percolation but the corresponding amplitude ratio is about 50 instead of 200.[22] Similar results were also observed for bond percolation of two-dimensional random lattice[19] in that the susceptibility exponent was found to be close to the lattice value but the amplitude ratio was found to be equal to about 14. In three-dimensional kinetic gelation,[21] the critical exponents are again estimated to be the same as for three-dimensional percolation, while the amplitude ratio R is apparently much smaller. More accurate estimates of amplitude ratios may be obtained by use of finite-size scaling; however, we did not concentrate on such calculations here.

In order to estimate the correlation-length exponent ν of continuum percolation for the PCS model, we have performed MCRG transformations of a $b\times b$ cell onto a $b'\times b'$ cell for various values of λ. The connectivity function $R(\eta,b)$ was obtained by MC simulation in a $b\times b$ cell for each selected b and λ. Plotted in Fig. 3 are the results for $\lambda=0.8$. Symbols are the simulation data and curves are the spline fits. Rescaling is performed for cells of $b=7$, 10, 15, 20, 25, 30, and 40 onto the cell of $b'=5$, and the fixed points $\eta^*_{b,b'}$ and the eigenvalues $\Lambda_{b,b'}$ of the recursion relation (3) are calculated from $R(\eta,b)$ and $R(\eta,b')$. The exponent ν is determined from the ln-ln plot of $\Lambda_{b,b'}$ versus b/b', and results for $\lambda=0$ and 0.8 are compared in Fig. 4. The slopes of the lines fitted from the data are 0.7517 ± 0.0126 and 0.7577 ± 0.0122 for $\lambda=0$ and 0.8, respectively. The corresponding correlation-length exponents are 1.33 ± 0.02 and 1.32 ± 0.02. The errors quoted were obtained from the linear regression but there may be additional statistical errors not accounted for. The result for $\lambda=0$ is in excellent agreement with the result of Vicsek and Kertesz[5] for randomly centered disks obtained by the same method. These values of ν are very close to each other and are also very close to the lattice value of $\frac{4}{3}$.[23] Similar transformations were also carried out for $\lambda=0.3$, 0.5, 0.7, and 0.9 and the results were found to lie between 1.30 and 1.35 depending on λ.

Considering these data and the value of γ for $\lambda=0.8$

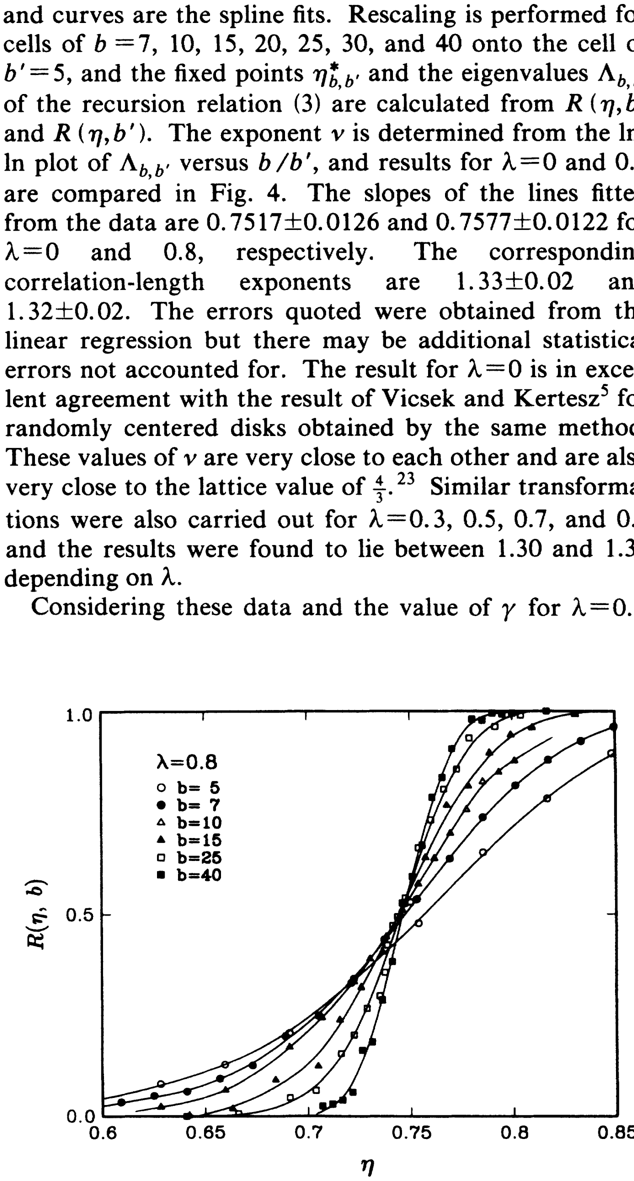

FIG. 3. Monte Carlo data for the connectivity function $R(\eta,b)$ for $\lambda=0.8$ as a function of η. The smooth curves through the data represent the best fits. Notice that as the cell size increases, $R(\eta,b)$ approaches a step function.

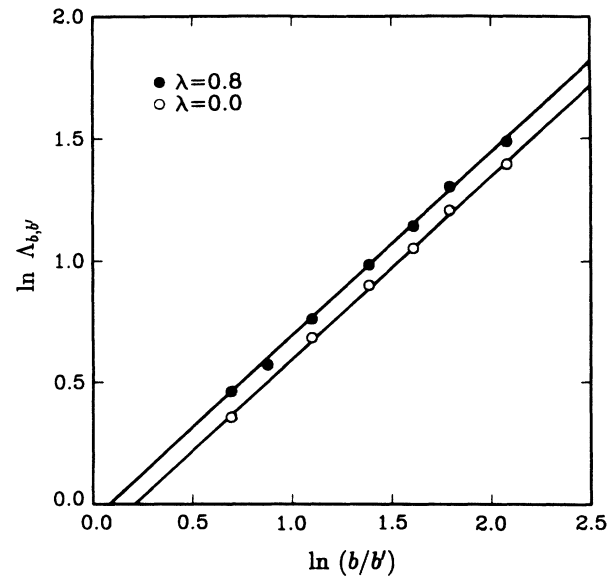

FIG. 4. Double natural logarithmic plot of $\Lambda_{b,b'}$ against b/b' for $\lambda=0$ and 0.8. Rescaling of cells of $b=7$, 10, 15, 20, 25, 30, and 40 was performed onto the cell of $b'=5$. The correlation-length exponents ν determined from the slopes of the best fits are $\nu=1.33\pm0.02$ and 1.32 ± 0.02 for $\lambda=0$ and 0.8, respectively.

obtained by MC simulation, we conclude that continuum percolation of the PCS model for $0<\lambda<1$ belongs to the same universality class as that of randomly centered disks and, thus, as that of ordinary lattice percolation, as far as the geometrical critical exponents are concerned.

The percolation thresholds were also estimated from the finite-size scaling analyses described by Eq. (8) for various values of λ between zero and unity. Here we have performed additional rescaling of cells of $b=10$, 15, 20, 25, 30, and 40 onto a cell of $b'=6$. As b/b' becomes infinite, the fixed points $\eta^*_{b,b'}$ for $b'=5$ and $b'=6$ are expected to approach a single point, η_c, following Eq. (8). Figure 5(a) shows the plots of $(b/b')^{-1/\nu}$ versus $\eta^*_{b,b'}$ for $\lambda=0$, i.e., randomly centered disks. The percolation threshold obtained from the intercept on the abscissa in this plot is $\eta_c=1.1283\pm0.0011$. The corresponding critical area fraction of inclusions is, if converted from the exact relation $\phi=1-\exp(-\eta)$, $\phi_c=0.6764\pm0.0010$. This value of ϕ_c is very close to the previous estimates $\phi_c=0.68$,[24] 0.67,[25] and 0.676.[4] It is, however, slightly smaller than that of recent work of this kind by Vicsek and Kertesz, $\phi_c=0.688\pm0.005$;[5] the difference might be due to different approximations[26] used to calculate the connectivity function. Plotted in Fig. 5(b) are corresponding results for $\lambda=0.8$. The intercept on the abscissa, i.e., critical reduced number density, is $\eta_c=0.7533\pm0.0005$. The corresponding area fraction ϕ_c was estimated using our recent numerical study of ϕ as a function of η;[13] $\phi_c=0.7142\pm0.0010$. The percolation thresholds η_c and ϕ_c for other values of λ are also obtained using the aforementioned technique and are listed in Table I.

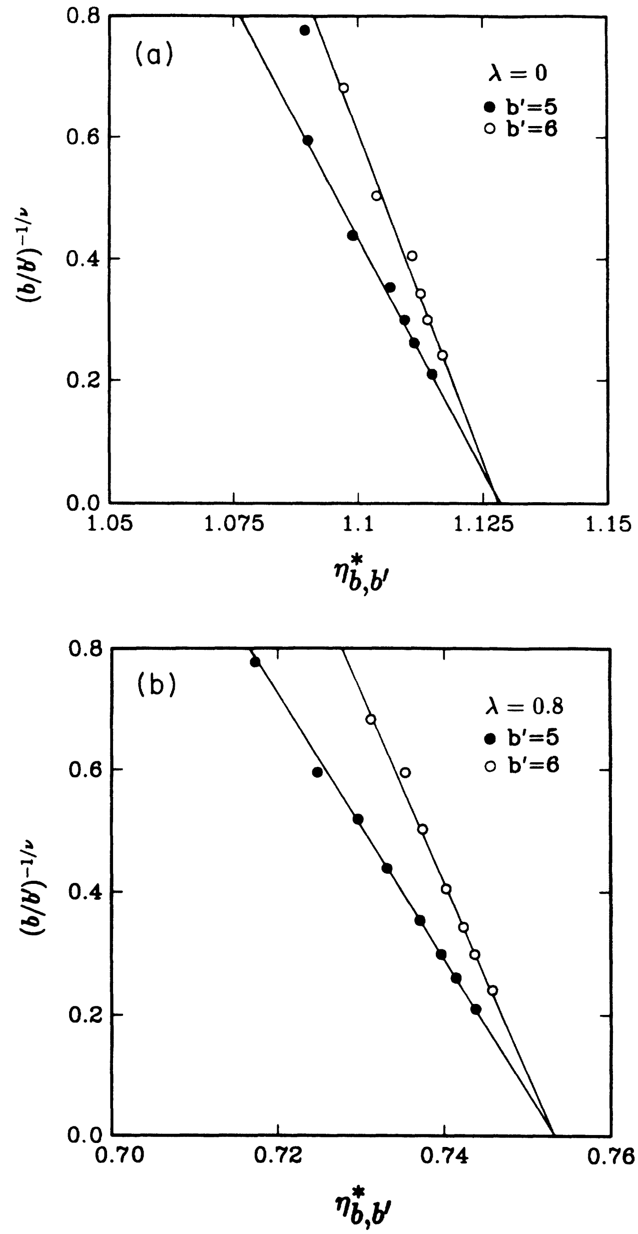

FIG. 5. Plots of the fixed points $\eta^*_{b,b'}$ against $(b/b')^{-1/\nu}$ for (a) $\lambda=0$ and (b) $\lambda=0.8$. Rescaling of cells of $b=7$, 10, 15, 20, 25, 30, and 40 was performed onto the cell of $b'=5$ (closed circles) and that of $b'=6$ (open circles). The percolation points η_c obtained from the intercepts on the abscissa are (a) 1.1283 ± 0.0011 and (b) 0.7533 ± 0.0005 for $\lambda=0$ and 0.8, respectively.

Our results for η_c are close to but appear to be slightly smaller than recent MC estimates by Bug, Safran, and Grest[9] obtained without making any type of extrapolations to the infinite-system limit. These discrepancies are expected since the percolation points in their work were approximated from the "fixed" points in a finite system according to a certain rule and such points are in general greater or smaller than true percolation points depending on the spanning rules employed.

TABLE I. Percolation thresholds for the PCS model obtained by the MCRG method for selected values of λ. The values of the critical area fraction ϕ_c were estimated from η_c using the recent numerical work of Lee and Torquato (Ref. 13). Errors listed are associated with linear regression in the finite-size scaling.

λ	η_c	ϕ_c
0.0	1.1283±0.0011	0.6764
0.3	0.9833±0.0013	0.6780
0.5	0.8520±0.0006	0.6797
0.7	0.7691±0.0006	0.6951
0.8	0.7533±0.0005	0.7142
0.9	0.7745±0.0004	0.7606
1.0	0.82[a]	0.82

[a] Hard-disk random close-packing area fraction is quoted from Ref. 27.

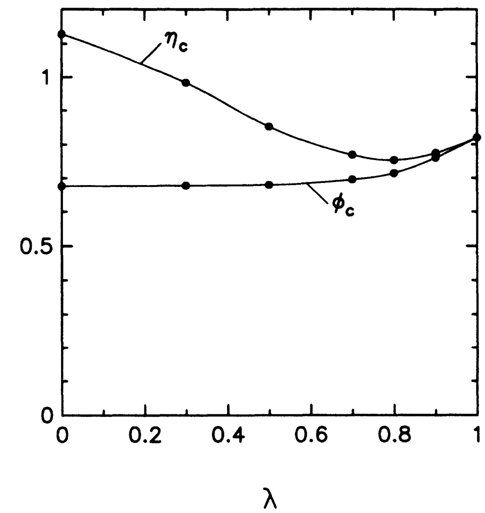

FIG. 6. Percolation thresholds as a function of the impenetrability parameter λ. Spline fits of the data for the critical reduced number density η_c and the critical area fraction ϕ_c.

In Fig. 6, the thresholds η_c and ϕ_c are plotted as a function of λ. The plot of η_c first decreases up to $\lambda=0.8$ and then increases toward the random close-packing area fraction of hard disks, $\phi_c=0.82$,[27] i.e., there is an optimum value of the excluded-volume parameter λ at which the number density required to percolate the system reaches a minimum. On the other hand, the critical area fraction ϕ_c is nearly constant up to λ equal to about 0.5 and thereafter it increases monotonically. This indicates that excluded-volume interactions among the particles do not change the critical area fraction significantly as long as the size of hard core is not very large. As λ is made larger than 0.5, repulsive interactions are such that the average coordination number decreases, making percolation more difficult, and thus the critical area fraction increases.

V. SUMMARY AND CONCLUSIONS

We have studied continuum percolation of the two-dimensional PCS model by MC simulations and MCRG approaches. The susceptibility exponent γ and correlation-length exponent ν were found to be close in value to those of randomly centered disks ($\lambda=0$) for some nonzero values of the impenetrability parameter λ. Thus we conclude that the two-dimensional PCS model for $0<\lambda<1$ belongs to the same universality class as that of randomly centered disks and ordinary lattice percolation, as far as geometrical critical exponents are concerned.

In this work, we restricted ourselves to one-parameter renormalizations. If one considers two-parameter renormalizations, such as that of η and λ, one can naturally expect from our results that there is a nontrivial fixed point on the $\lambda=0$ axis in the two-parameter space, with renormalization flows directed toward this "critical" point along the λ axis. We, however, do not know if there is any additional unstable fixed point on the $\lambda=1$ axis, i.e., for the case of hard disks. This situation is quite similar to other two-parameter renormalizations in critical phenomena, such as the renormalizations of random walks with an excluded-volume parameter[28] and that of ferromagnetic systems with impurities.[29] In the former case, rescalings were performed for the fugacity of each step of the random walks and the excluded-volume parameter. In the latter case, on the other hand, field-theoretical calculations resulted in a well-known heuristic argument, called the Harris criterion. In many such cases, it has been found that there are at least two nontrivial fixed points in the two extreme limits of the second parameter and that renormalization flows are from one to the other, indicating that critical behavior for intermediate values of the second parameter is the same as that of either one of these two "critical" points. If one accepts such analogies in the present problem also, one can expect another unstable fixed point on the line $\lambda=1$. The critical behavior at such a point, if any, should be different from that of randomly centered disks. Our present work does not rule out such possibilities.

We also reported that amplitude ratio of the susceptibility below and above the percolation threshold for $\lambda=0.8$ and $L=50$. This value of R was found to be significantly greater than that of randomly centered disks and that of the lattice value. Further investigations of this problem for other values of λ are in progress by one of us.[22]

Accurate determinations of η_c for the PCS model were also carried out for selected values of λ using finite-size scaling analyses. The critical area fractions ϕ_c are also tabulated for the first time as a function of λ.

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[1]D. Stauffer, Phys. Rep. **54**, 1 (1979); also see D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).

[2]J. W. Essam, Rep. Prog. Phys. **43**, 833 (1980).

[3]A. Coniglio, V. DeAngelis, and A. Forlani, J. Phys. A **10**, 1123 (1977).

[4]E. T. Gawlinski and H. E. Stanley, J. Phys. A **14**, L291 (1981).

[5]T. Vicsek and J. Kertesz, J. Phys. A **14**, L31 (1981).

[6]W. Klein, Phys. Rev. B **26**, 2677 (1982).

[7]E. T. Gawlinski and S. Redner, J. Phys. A **16**, 1063 (1983).

[8]Y. C. Chiew and E. D. Glandt, J. Phys. A **16**, 2599 (1983); T. DeSimone, R. M. Stratt, and S. Demoulini, Phys. Rev. Lett. **56**, 1140 (1986); J. Xu and G. Stell, J. Chem. Phys. **89**, 2344 (1988). These papers study the Ornstein-Zernike equation for the "direct connectedness" function.

[9]A. L. R. Bug, S. A. Safran, G. S. Grest, and I. Webman, Phys. Rev. Lett. **55**, 1896 (1985).

[10]E. M. Sevick, P. A. Monson, and J. M. Ottino, J. Chem. Phys. **88**, 1198 (1988).

[11]S. B. Lee and S. Torquato, J. Chem. Phys. **89**, 6427 (1988).

[12]S. Torquato, J. Chem. Phys. **91**, 5079 (1984); **83**, 4776 (1985); **84**, 6345 (1986).

[13]P. A. Rikvold and G. Stell, J. Colloid. Interface Sci. **108**, 158 (1985); S. B. Lee and S. Torquato, J. Chem. Phys. **89**, 3258 (1988).

[14]N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953); see also W. W. Woods, in *Physics of Simple Liquids,* edited by H. N. V. Temperly (North-Holland, Amsterdam, 1968).

[15]P. J. Reynolds, H. E. Stanley, and W. Klein, Phys. Rev. B **21**, 1223 (1980).

[16]F. Lado, J. Chem. Phys. **49**, 3092 (1968) and references therein.

[17]J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3438 (1976).

[18]M. F. Sykes, D. S. Gaunt, and M. Glen, J. Phys. A **9**, 97 (1976).

[19]D. Y. Kim, H. J. Hermann, and D. P. Landau, Phys. Rev. B **35**, 3661 (1987).

[20]I. Balberg, Phys. Rev. B **37**, 2391 (1988).

[21]H. J. Hermann and D. P. Landau, Phys. Rev. Lett. **49**, 412 (1982).

[22]Gawlinski and Stanley calculated the mean cluster size defined by $S=\sum_s' n_s s^2/\sum_s' n_s s$. In this definition, however, since it is known that there is a singularity in the denominator for $\eta > \eta_c$ (see, e.g., Ref. 1), it is not clear that amplitude ratio defined in their works is the same as what we define in the present work. In fact, one of us (S.B.L.), in work to be published, obtained $R \cong 192$ by Monte Carlo simulations using Eq. (9).

[23]M. P. M. den Nijs, J. Phys. A **12**, 1857 (1979).

[24]G. E. Pike and C. H. Seager, Phys. Rev. B **10**, 1421 (1974).

[25]D. H. Fremlin, J. Phys. (Paris) **37**, 813 (1976).

[26]Vicsek and Kertesz obtained $R(\phi_2,b)$ via an integral of the probability density function $L(\phi_2,b)$. Particle centers are distributed randomly in a given cell and critical radius of particles r_c was found for which system begins to percolate. The corresponding area fraction was calculated from $\phi_2=1-\exp(-\rho\pi r_c^2)$, and the distribution of ϕ_2, i.e., $L(\phi_2,b)$, was calculated over many realizations. In this formulation, however, since r_c varies from realization to realization, the relative size of the system b/r_c is not constant over the realizations for each b.

[27]J. G. Berryman, Phys. Rev. A **27**, 1053 (1983).

[28]F. Family and H. Gould, J. Chem. Phys. **80**, 3892 (1984).

[29]A. B. Harris, J. Phys. C **7**, 1671 (1974).