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ON THE USE OF HOMOGENIZATION THEORY TO DESIGN OPTIMAL PIEZOCOMPOSITES FOR HYDROPHONE APPLICATIONS

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ABSTRACT

We consider an optimal design of composite hydrophones consisting of parallel piezoelectric PZT rods that are embedded in a porous polymer matrix. Given the material properties of the polymer and PZT ceramic, we have optimally designed the piezocomposite to maximize the hydrostatic coupling factor, hydrophone figure of merit, or electromechanical coupling factor, using the methods of homogenization theory. The optimal composite is obtained by using a two-step procedure : (i) first we find the ideal structure of the matrix material by weakening the polymer by an optimal arrangement of pores, and (ii) then we embed the PZT rods in this matrix. The design parameters are the shape, volume fraction, and spatial arrangement of the piezoceramic rods, and the structure of the matrix material. It turns out that the optimal matrix is highly anisotropic and is characterized by negative Poisson's ratios in certain directions. The optimal composites possess performance characteristics that are significantly higher than those of a piezocomposite with an isotropic polymer matrix. The results can be viewed as theoretical upper bounds on the hydrophone performance. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Piezoelectric transducers have been employed as sensors and transmitters of acoustic signals in medical imaging, non-destructive testing, and hydrophones. This paper is concerned with optimally designing the performance characteristics of composite hydrophones consisting of parallel piezoceramic PZT rods that are embedded in a porous polymer matrix. The hydrophone is assumed to operate in the low-frequency range (i.e. the wavelength of the pressure signal is much larger than spacing between the PZT rods) and hence its behavior can be described in the quasistatic limit.

One may ask why one would want to make a composite to begin with or, in other words, why is pure piezoceramic not used since it is the only material with piezoelectric properties? The basic problem is that under hydrostatic load, the anisotropic piezoelectric response of pure PZT is such that it has poor hydrophone performance characteristics. Specifically, consider a PZT rod poled in the axial direction (x_3-
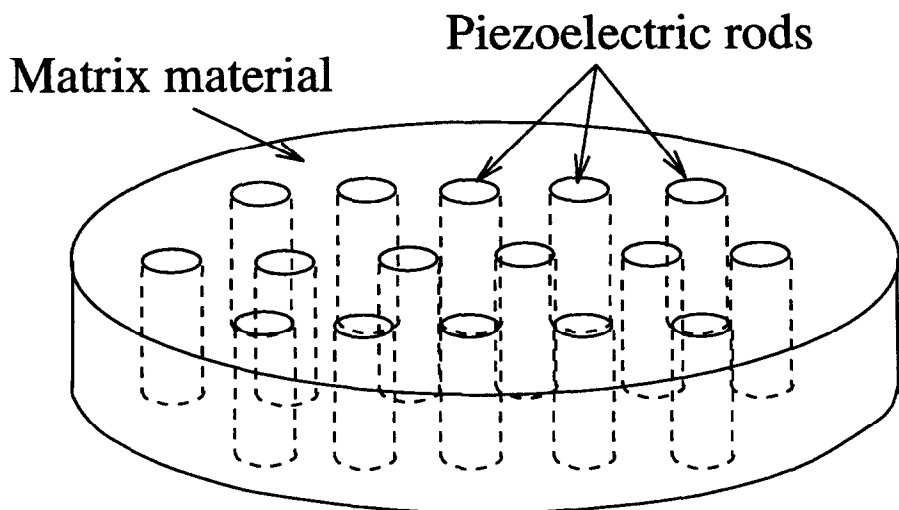
Piezoelectric rods

Matrix material

Fig. 1. Schematic of a 1-3 piezocomposite.

direction) subjected to hydrostatic load. The induced polarization field in the axial direction is found to be proportional to applied pressure, i.e.

$$D_3 = d_h T, \quad d_h = d_{33} + 2d_{13}, \tag{1.1}$$

where D_3 is the dielectric displacement in the x_3-direction, T is the amplitude of the applied pressure, d_h is the hydrostatic coupling coefficient, d_{33} and d_{13} are the longitudinal and transverse piezoelectric coefficients characterizing the dielectric response for axial and lateral compression, respectively. Unfortunately, d_{33} and d_{13} have opposite signs, thus resulting in a relatively small hydrostatic coupling factor d_h. For example, for pure piezoceramic PZT5A, $d_{33} = 374$ pC/N and $d_{13} = -171$ pC/N. Therefore, $d_h = 32$ pC/N which is small compared to d_{33}.

We note that there are other sensitivity measures besides d_h. The *voltage coefficient* $g_h = d_h/\varepsilon_{33}$ (where ε_{33} is dielectric constant in the x_3-direction), the *hydrophone figure of merit* $d_h g_h$, and the *electromechanical coupling factor* k_h are examples.

Experiments for specific polymer/ceramic systems show that composites with high hydrophone sensitivity can be achieved by combining the piezoceramic rods and a soft polymer matrix (see, e.g. Klicker *et al.*, 1981; Newnham and Ruschau, 1991; Ting *et al.* 1990). Figure 1 schematically depicts such a "1–3 piezocomposite". An appropriately designed piezocomposite is capable of converting an applied hydrostatic field into a predominantly axial stress on the rods, thus enhancing all of the hydrophone characteristics. Using simple models in which the elastic and electric fields were taken to be uniform in the difference phases, Haun and Newnham (1986), Chan and Unsworth (1989) and Smith (1991, 1993) qualitatively explained the enhancement due to the Poisson's ratio effect.

The basic physics behind such a device is the following: the load applied to the piezocomposite in the axial direction is taken almost entirely by the piezoelectric rods, if the polymer is soft (compared with the piezoelectric rods). Therefore, the coefficient

d_{33}^* of the composite will be approximately equal to that of the pure piezoceramic, d_{33}, if the stiffness of the polymer matrix is small compared with the stiffness of ceramic rods. On the other hand, the transverse load will result in a transverse stress on the rods which is equal to the applied load. Therefore, the effective d_{13}^* coefficient of the composite will have value close to fd_{13}, where f is the volume fraction of the rods. Thus, the effective hydrostatic coupling factor is approximately given by

$$d_h^* = d_{33} + 2fd_{13}. \qquad (1.2)$$

For example, for a composite consisting of PZT5A with $f = 0.1$, this simple analysis yields $d_h^* = 340$ pC/N.

Equation (1.2) shows that the volume fraction of the rods should be small to achieve enhancement of the hydrophone characteristics. However, one cannot let the volume fraction of the rods f be very small, because in this case the axial load will be taken by the polymer rather than the intended piezoceramic. Therefore, among other design parameters, there exists some optimal volume fraction f that maximizes the hydrostatic coupling factor d_h^*.

Smith (1991) was the first to propose that even greater enhancement in hydrophone characteristics can be achieved by using matrices with negative Poisson's ratio. Indeed, for such a composite the transverse load on the polymer matrix will lead to a contraction of the polymer in the axial direction, thus increasing the axial load on the piezoceramic rods. This results in enhancement of the hydrophone performance. A more sophisticated analysis has been recently given by Avellaneda and Swart (1994) using the so-called differential-effective-medium approximation.

It was found that the performance of the piezocomposite depends significantly on the properties and the volume fraction of the rods, and on the mechanical properties of the polymer matrix. For example, the use of a matrix with negative Poisson's ratio or a porous matrix increases the sensitivity of the hydrophone by an order of magnitude. Such polymer foams with negative Poisson's ratio were manufactured by Lakes (1987).

It is our aim to find optimal composites that maximize the hydrophone characteristics. This is accomplished using the methods of homogenization theory by extending the analyses of Avellaneda and Swart (1994). We assume that the properties of the PZT ceramic are given and optimize the composite over the shape, arrangement, and volume fraction of the PZT rods. However, our main contribution is that we depart from the assumption of isotropy of the matrix, and require only transverse isotropy of this material. We treat the matrix material itself as a composite; it is assumed to be made of a polymer with given properties, weakened by an optimal arrangement of pores. The microstructure of the matrix material is an additional control in the problem that we study. As we will see, the optimal matrix material is highly anisotropic, with a large ratio of the minimal and maximal eigenvalues of the stiffness tensor.

Theoretical results obtained in this paper are in agreement with numerical experiments performed by Sigmund et al. (1996). By using a topology optimization procedure, these authors have designed the microstructures of porous polymer matrices to optimize performance characteristics of the piezocomposites for hydrophone applications.

The rest of the paper is organized as follows: in Section 2, we give a brief summary of the formulas that describe processes in piezoelectric composites and performance characteristics of hydrophones. In Section 3, we discuss the design parameters of the problem, i.e., the shape, arrangement, and volume fraction of the PZT rods, and also the microstructure of the porous-polymer anisotropic material that forms the matrix of the PZT-matrix composite. In Section 4, we analyze the problem and transform the design variables in a form convenient for our analyses. Section 5 presents the results of numerical optimization. In Section 6 we study sensitivity of the performance characteristics to variations of the design variables. In Section 7 we describe the effective properties of optimal matrix materials. Section 8 summarizes the results of the paper.

2. STATE EQUATIONS FOR PIEZOELECTRIC MATERIALS

In this section we give a brief summary of the formulas that describe piezoelectric hydrophones [see, e.g. Smith, 1991, 1993; Avellaneda and Swart, 1994]. We start with the basic equations of piezoelectricity. For low-frequency oscillations (i.e., in the quasistatic approximation), the elasticity equations and Maxwell's equations reduce to

$$\nabla \cdot \mathbf{T} = 0, \quad \mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t), \tag{2.1}$$

and

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = 0, \tag{2.2}$$

respectively. Here $\mathbf{T}$ and $\mathbf{S}$ are the stress and strain tensors, $\mathbf{u}$ is the displacement vector, $\mathbf{D}$ is the dielectric displacement, $\mathbf{E}$ is the electrical field, and superscript t denotes transponded tensor, i.e.,

$$(\mathbf{a}^t)_{ij} = a_{ji}, \quad (\mathbf{d}^t)_{kij} = d_{jik}. \tag{2.3}$$

These fields are coupled through constitutive relations of piezoelectricity, i.e.

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{s}^E & \mathbf{d} \\ \mathbf{d}^t & \varepsilon^T \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{E} \end{pmatrix}, \tag{2.4}$$

where $\mathbf{s}^E = s^E_{ijkl}$ is a fourth order compliance tensor under short circuit boundary conditions, $\mathbf{d} = d_{ijk}$ is a third order piezoelectric stress coupling tensor and $\varepsilon^T = \varepsilon^T_{ij}$ is the second order free-body dielectric tensor. An alternative form of the same constitutive relations is

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{c}^E & -\mathbf{e} \\ \mathbf{e}^t & \varepsilon^S \end{pmatrix} \begin{pmatrix} \mathbf{S} \\ \mathbf{E} \end{pmatrix}, \tag{2.5}$$

where $\mathbf{c}^E = (\mathbf{s}^E)^{-1}$ is the short-circuit stiffness tensor, $\varepsilon^S = \varepsilon^T - \mathbf{d}^t(\mathbf{s}^E)^{-1}\mathbf{d}$ is a clamped-body dielectric tensor, and $\mathbf{e} = (\mathbf{s}^E)^{-1}\mathbf{d}$ is the piezoelectric strain tensor.

The object under study is a composite of PZT-ceramic rods in a porous polymer. If the wavelength of the applied field is much larger than the spacing between rods, the behavior of a composite can be characterized by the averaged equations

$$
\begin{pmatrix} \langle \mathbf{S} \rangle \\ \langle \mathbf{D} \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{s}_*^{\mathrm{E}} & \mathbf{d}_* \\ \mathbf{d}_*^{\mathrm{t}} & \boldsymbol{\varepsilon}_*^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \langle \mathbf{T} \rangle \\ \langle \mathbf{E} \rangle \end{pmatrix}, \tag{2.6}
$$

where the angular brackets denote volume averaging and the index "*" refers to the effective properties. Similar to relation (2.5), one can write the averaged constitutive relations as

$$
\begin{pmatrix} \langle \mathbf{T} \rangle \\ \langle \mathbf{D} \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{c}_*^{\mathrm{E}} & -\mathbf{e}_* \\ \mathbf{e}_*^{\mathrm{t}} & \boldsymbol{\varepsilon}_*^{\mathrm{S}} \end{pmatrix} \begin{pmatrix} \langle \mathbf{S} \rangle \\ \langle \mathbf{E} \rangle \end{pmatrix}, \tag{2.7}
$$

where

$$
\mathbf{c}_*^{\mathrm{E}} = (\mathbf{s}^{\mathrm{E}})_*^{-1}, \quad \boldsymbol{\varepsilon}_*^{\mathrm{S}} = \boldsymbol{\varepsilon}_*^{\mathrm{T}} - \mathbf{d}_*^{\mathrm{t}}(\mathbf{s}_*^{\mathrm{E}})^{-1}\mathbf{d}_*, \quad \mathbf{e}_* = (\mathbf{s}_*^{\mathrm{E}})^{-1}\mathbf{d}_*. \tag{2.8}
$$

In what follows we will omit the index "*" in the notation for the effective properties, use the index r for the properties of the PZT ceramic rods, and use index m for the matrix material.

We employ dyadic notation for the problem under study, i.e.

$$
d_{13} = d_{113}, \quad d_{33} = d_{333}, \quad s_{ij} = s_{iijj}, \quad i,j = 1,2,3, \tag{2.9}
$$

etc., where the coefficient on the right hand sides of (2.9) are the coefficients of the corresponding tensors in the Cartesian basis. We will assume that the composite is transversely isotropic and hence we have

$$
\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ D_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & -e_{13} \\ c_{12} & c_{11} & c_{13} & -e_{13} \\ c_{13} & c_{13} & c_{33} & -e_{33} \\ e_{13} & e_{13} & e_{33} & -\varepsilon_{33} \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ E_3 \end{bmatrix}, \tag{2.10}
$$

where c_{ij}, e_{ij}, and ε_{ij} are the dyadic coefficients of the tensors $\mathbf{c}^{\mathrm{E}}$, $\mathbf{e}$, and $\boldsymbol{\varepsilon}^{\mathrm{S}}$, respectively. Here we have omitted the uncoupled part of the system that is not important for our purposes.

The response of the transversely isotropic hydrophone composite under hydrostatic pressure $\langle \mathbf{T} \rangle = T\delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise) is commonly characterized by three quantities:

(i) The hydrostatic coupling coefficient defined by

$$
d_{\mathrm{h}} = \langle D_3 \rangle / T = d_{33} + 2d_{13} \tag{2.11}
$$

   measures the polarization sensitivity.
(ii) The hydrophone figure of merit

$$
d_{\mathrm{h}}g_{\mathrm{h}} = d_{\mathrm{h}}^2 / \varepsilon_{33}^{\mathrm{T}} \tag{2.12}
$$

   is another measure of the sensitivity of the composite.

(iii) The electromechanical coupling factor, k_h, defined by

$$k_h = \sqrt{\frac{d_h^2}{\varepsilon_{33}^T s_h}}, \tag{2.13}$$

where k_h^2 measures the overall acoustic/electric power conversion. Here

$$s_h = 2s_{11} + 2s_{12} + 4s_{13} + s_{33} \tag{2.14}$$

is a dilatational compliance and s_{ij} are the dyadic coefficients of the tensor $\mathbf{s}^E$.
These performance characteristics can be written in the following way

$$d_h = \mathbf{v} \cdot C^{-1} \cdot \mathbf{e}, \quad d_h g_h = \frac{(\mathbf{v} \cdot C^{-1} \cdot \mathbf{e})^2}{\varepsilon_{33} + \mathbf{e} \cdot C^{-1} \cdot \mathbf{e}}, \quad k_h^2 = \frac{(\mathbf{v} \cdot C^{-1} \cdot \mathbf{e})^2}{(\varepsilon_{33} + \mathbf{e} \cdot C^{-1} \cdot \mathbf{e})\mathbf{v} \cdot C^{-1} \cdot \mathbf{v}}$$

$$\tag{2.15}$$

where

$$C = \begin{pmatrix} K & c_{13} \\ c_{13} & c_{33} \end{pmatrix}, \quad \mathbf{e} = (e_{13}\ e_{33}), \quad \mathbf{v} = (1\ 1), \tag{2.16}$$

and $K = (c_{11} + c_{12})/2$ is the transverse bulk modulus. Recall that the coefficients in (2.15) and (2.16) refer to the effective characteristics.

One can express the effective coefficients of the composite (those that are relevant for our analyses) in terms of the coefficients of the PZT-ceramics, polymer matrix, and only the structural parameter p (Avellaneda and Swart, 1994):

$$c_{13} = c_{13}^m + fp(c_{13}^r - c_{13}^m),$$

$$e_{13} = e_{13}^m + fp(e_{13}^r - e_{13}^m),$$

$$c_{33} = c_{33}^m + f\left\{c_{33}^r - c_{33}^m + (p-1)\frac{(c_{13}^r - c_{13}^m)^2}{(K^r - K^m)}\right\},$$

$$e_{33} = e_{33}^m + f\left\{e_{33}^r - e_{33}^m + (p-1)\frac{(c_{13}^r - c_{13}^m)(e_{13}^r - e_{13}^m)}{(K^r - K^m)}\right\},$$

$$\varepsilon_{33} = \varepsilon_{33}^m + f\left\{\varepsilon_{33}^r - \varepsilon_{33}^m - (p-1)\frac{(e_{13}^r - e_{13}^m)^2}{(K^r - K^m)}\right\}, \tag{2.17}$$

where coefficients with the superscripts r and m denote the properties of the rod and the matrix, respectively, and f is the volume fractions of the PZT rods. The parameter p is related to the effective transverse bulk modulus K of the composite via the relation

$$p = \frac{1}{f} \cdot \frac{K - K^m}{K^r - K^m}, \tag{2.18}$$

where $K^m = (c_{11}^m + c_{12}^m)/2$, $K^r = (c_{11}^r + c_{12}^r)/2$. There are other effective coefficients but they have no importance for hydrophone applications.

3. DESIGN PARAMETERS

In this section we discuss all of the design parameters of the problem, the restrictions on these parameters, and the properties of the original phases.

3.1. *Volume fraction of the PZT rods*

All of the properties of the composite are very sensitive to the volume fraction f of the PZT rods and hence it is one of the main design variables of the problem. Previous studies suggested that the optimal volume fraction is small and should lie in the interval $f \in [0.05, 0.20]$. As we will see, a similar conclusion remains valid for the piezoelectric composite with transversely isotropic matrix.

3.2. *Arrangement of the PZT rods*

As was mentioned in Section 2, given the rods and the matrix properties and volume fractions, the hydrophone characteristics are uniquely defined by the effective transverse bulk modulus K that depends on the spatial arrangement of the PZT rods. For any arrangement of rods, this modulus must satisfy the Hashin–Shtrikman bulk modulus bounds that for the plane problem were give by Hashin (1965):

$$K^- \leqslant K \leqslant K^+, \tag{3.1}$$

where

$$K^- = fK^r + (1-f)K^m - \frac{f(1-f)(K^m - K^r)^2}{fK^m + (1-f)K^r + \mu^m}, \tag{3.2}$$

$$K^+ = fK^r + (1-f)K^m - \frac{f(1-f)(K^m - K^r)^2}{fK^m + (1-f)K^r + \mu^r}. \tag{3.3}$$

Here $\mu^m = (c_{11}^m - c_{12}^m)/2$ and $\mu^r = (c_{11}^r - c_{12}^r)/2$, $(\mu^m \leqslant \mu^r)$ are the transverse shear moduli of the matrix and PZT rods, respectively. Note that there exist composites [Hashin, 1965, coated-cylinders assemblages] that correspond to the lower and upper Hashin–Shtrikman bounds (3.2) and (3.3).

Therefore, the influence of shape and distribution of the rods on the hydrophone characteristics can be uniquely determined by the dimensionless parameter δ as follows

$$K = (1-\delta)K^- + \delta K^+. \tag{3.4}$$

The value $\delta = 0$ corresponds to the Hashin–Shtrikman bulk modulus lower bound and the value $\delta = 1$ corresponds to the Hashin–Shtrikman upper bound. The advantage of such a parameter is that the range of its variation does not depend on volume fraction. One can treat δ as an independent design variable that completely defines the microstructure of the composite. Note that

$$p = \frac{\delta(1-f)(K^r - K^m)(\mu^r - \mu^m) + (K^m + \mu^m)(fK^m + (1-f)K^r + \mu^r)}{(fK^m + (1-f)K^r + \mu^m)(fK^m + (1-f)K^r + \mu^r)}. \tag{3.5}$$

We do not prescribe the shape of the rods' cross-section, but only require that the

geometry of the composite does not depend on the vertical co-ordinate (direction along the rods). Note also that (3.5) is the only place in our analyses that involves the shear moduli μ^m and μ^r.

3.3. *Properties of the matrix*

The hydrophone characteristics are very sensitive to the properties of the matrix material. For example, Avellaneda and Swart (1994) showed that introducing additional porosity into the polymer matrix may dramatically improve the performance of the composite. They also showed that decreasing the Poisson's ratio of the matrix may result in enhanced performance of the composite.

Here we further explore the idea of optimally designing the matrix for the piezoelectric composite. We depart from the assumption of isotropy of the matrix material and use a transversely isotropic matrix which itself is treated as a composite. Thus, the optimal hydrophone composite design is a two-step process: first, we create an optimal matrix by weakening the polymer by an optimal arrangement of pores. Then we embed the PZT rods into this matrix. In this section, we discuss the first step of this process.

We assume that the matrix material is comprised of an isotropic polymer phase with a given stiffness tensor $\mathbf{c}_p$ and a void phase (with zero stiffness). Hence, it can be viewed as an isotropic polymer which is weakened by an optimal arrangement of pores. Milton and Cherkaev (1995a,b) raised the question as to whether the trivial bounds on the effective stiffness tensor $\mathbf{c}_m$ of a composite of voids and a phase with the properties of $\mathbf{c}_p$, namely,

$$0 \leqslant \mathbf{c}_m \leqslant \mathbf{c}_p, \tag{3.6}$$

are in fact realizable. If these trivial bounds are realizable then it follows that there exist optimal arrangements of the pores that lead to any of the tensors $\mathbf{c}_m$ satisfying inequalities (3.6). Let us assume that this is the case. By the tensor inequality of the type $\mathbf{c}_p \geqslant \mathbf{c}_m$, we shall mean that the difference $\mathbf{c}_p - \mathbf{c}_m$ is positive semi-definite matrix.

We shall assume that $\mathbf{c}_m$ is transversely isotropic. Then the inequalities (3.6) lead to the following restrictions

$$0 \leqslant \begin{bmatrix} c^m_{11} & c^m_{12} & c^m_{13} \\ c^m_{12} & c^m_{11} & c^m_{13} \\ c^m_{13} & c^m_{13} & c^m_{33} \end{bmatrix} \leqslant \begin{bmatrix} c^p_{11} & c^p_{12} & c^p_{12} \\ c^p_{12} & c^p_{11} & c^p_{12} \\ c^p_{12} & c^p_{12} & c^p_{11} \end{bmatrix} \tag{3.7}$$

on the parameters c^m_{11}, c^m_{12}, c^m_{13}, c^m_{33}, that characterize the elastic properties of the matrix. One can verify that the matrix inequality (3.7) can be reduced to the following two simpler conditions:

$$0 \leqslant \begin{pmatrix} K^m & c^m_{13} \\ c^m_{13} & c^m_{33} \end{pmatrix} \leqslant \begin{pmatrix} K^p & c^p_{12} \\ c^p_{12} & c^p_{11} \end{pmatrix} \tag{3.8}$$

and

$$0 \leqslant \mu^m \leqslant \mu^p. \tag{3.9}$$

Here $\mu^m = (c_{11}^m - c_{12}^m)/2$ and $\mu^p = (c_{11}^p - c_{12}^p)/2$ are the transverse shear moduli of the matrix and the polymer, respectively.

Although there is no theoretical lower bound for the stiffness of the porous matrix material, it is unrealistic to expect zero stiffness of the hydrophones to be convenient for applications and hence we place restrictions on the matrix c_m via the inequalities

$$\mu^m \geqslant a\mu^p \tag{3.10}$$

and

$$\begin{pmatrix} K^m & c_{13}^m \\ c_{13}^m & c_{33}^m \end{pmatrix} \geqslant a\mu^m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.11}$$

where μ^p is the shear modulus of the polymer, and $a < 1$ is given. The exact form of the restrictions (3.10) and (3.11) is not crucial; we choose the form that is the most convenient for us. However, the performance characteristics are extremely sensitive to the value of the parameter a. In our numerical experiments we assume that $a = 0.03$. Decreasing a may lead to an even more dramatic increase of the values d_h, $d_h g_h$ and k_h. We illustrate this dependence in Section 6.

Let us now discuss dielectric properties of the matrix material. We will assume that the dielectric constant ε_{33}^m of the matrix material is equal to the dielectric constant ε_{33}^p of the polymer. This turns out to be a reasonable assumption. Indeed, the dielectric constant of the polymer is only 3.5 larger than that for vacuum, $\varepsilon_{33}^p/\varepsilon_0 = 3.5$, but is much smaller than the dielectric constant of PZT, $(\varepsilon_{33}^T)^r/\varepsilon_{33}^p = 486$. Therefore, the effective dielectric constant ε_{33}^T of the composite will be mainly determined by the dielectric constant $(\varepsilon_{33}^T)^r$ of the PZT ceramic. Numerically we found that even a variation of ε_{33}^m between the value for the polymer and that for the void space leads to only 6% variation in the values of $d_h g_h$ and $k_h (d_h$ does not depend on the dielectric properties).

In summary, the design parameters of the matrix include the three parameters $K^m = (c_{11}^m + c_{12}^m)/2$, c_{13}^m, and c_{33}^m [influencing the hydrophone performance directly through (2.17) and (2.18)] and a fourth parameter $\mu^m = (c_{11}^m - c_{12}^m)/2$ which enters the problem through (3.2)–(3.4). These four parameters are subjected to restrictions (3.8)–(3.11).

3.4. Properties of the piezoceramic and polymer

We assume that the moduli of the piezoceramic rods (indicated by the superscript r) and the polymer (indicated by the superscript p) are given as follows:

$$s_{11}^r = 16.4, \quad s_{12}^r = -5.74, \quad s_{13}^r = -7.22, \quad s_{33}^r = 18.8,$$

$$d_{13}^r = -171, \quad d_{33}^r = 374, \quad (e_{33}^T)^r = 1700\varepsilon_0, \tag{3.12}$$

$$s_{11}^p = s_{33}^p = 400, \quad s_{12}^p = s_{13}^p = -148, \quad d_{13}^p = d_{33}^p = 0, \quad (e_{33}^T)^p = 3.5\varepsilon_0,$$

$$\tag{3.13}$$

where s_{ij}^r, d_{ij}^r, and ε_{ij}^r are dyadic coefficients of the tensors $(s^E)^r$, $(d)^r$, and $(\varepsilon^T)^r$ that describe the PZT piezoceramic properties, and corresponding coefficients with the

superscript p describing the properties of the polyurethane polymer. Here the s-coefficients are measured in the units $10^{-12}\,\text{m}^2/\text{N} = \mu\text{m}^2/\text{N}$, e-coefficients are measured in the units $10^{-12}\,\text{C/N} = \text{pC/N}$, and

$$\varepsilon_0 = \frac{1}{4\pi}\frac{10^{-9}}{8.98755}\frac{C^2}{\text{Nm}^2} = \frac{1}{4\pi}\frac{10^3}{8.98755}\frac{pC^2}{\text{N}\mu\text{m}^2} \tag{3.14}$$

is the dielectric constant of the vacuum. By using (3.12) and (3.13), one can obtain the coefficients of the corresponding tensors $\mathbf{c}^E$, $\mathbf{e}$, and $\boldsymbol{\varepsilon}^S$ that are used to calculate the effective properties of the piezocomposite.

4. OPTIMAL DESIGN PROBLEM

We are now in a position to formulate the optimal design problem: given the parameters of the PZT ceramic (3.12) and the polyurethane polymer (3.13), optimize the matrix moduli K^m, μ^m, c_{13}^m, and c_{33}^m [subject to the restrictions (3.8)–(3.11)], the structural parameter $\delta \in [0, 1]$, and the volume fraction $f \in [0, 1]$ of the PZT rods in order to achieve the best performance of the hydrophone. Hence, since all of the performance characteristics (2.15) might be important for applications, we will find optimal designs that maximise each of the mentioned criteria, or maximises a weighted sum of them.

We found that the initially chosen optimization parameters (the coefficients of the $\mathbf{c}^m$ matrix) are not convenient to analyze the results. Therefore, we choose an alternative but equivalent set of controls, namely, the eigenvalues λ_1, λ_2 of the matrix

$$\begin{pmatrix} K_m & c_{13}^m \\ c_{13}^m & c_{33}^m \end{pmatrix} \tag{4.1}$$

and the scalar parameter $x \in (-\infty, \infty)$ that describes the direction of the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of this matrix as follows

$$\mathbf{v}_1 = (-1/x \quad 1), \quad \mathbf{v}_2 = (x \quad 1). \tag{4.2}$$

The coefficients K^m, c_{13}^m, and c_{33}^m are equal

$$K^m = \frac{\lambda_1 + x^2\lambda_2}{1+x^2}, \quad c_{13}^m = -\frac{(\lambda_1 - \lambda_2)x}{1+x^2}, \quad c_{33}^m = \frac{\lambda_1 x^2 + \lambda_2}{1+x_2}, \tag{4.3}$$

in terms of these new controls.

To summarize, the optimal design problems that we address can be reformulated as follows: Find the set of values of the parameters f, δ, μ^m, λ_1, λ_2, and x that maximize each of the following functionals:

(i) absolute value of the hydrostatic piezoelectric coefficient $|d_h|$;
(ii) hydrophone figure of merit $d_h g_h$;
(iii) hydrostatic electromechanical coupling factor k_h;
(iv) combination of the parameters $|d_h|$, $d_h g_h$, and k_h;

We will not precisely specify the last functional but will simply try to find the set of the design variables that keep the values of d_h, $d_h g_h$, and k_h within equal distance (15–20%) of their maximal possible values.

It is helpful to present the expressions for the coefficients of the stiffness matrix c_m in terms of our new design parameters. One can check that

$$c_{11}^m = \frac{\lambda_1 + x^2 \lambda_2}{1 + x^2} + \mu^m, \quad c_{12}^m = \frac{\lambda_1 + x^2 \lambda_2}{1 + x^2} - \mu^m, \tag{4.4}$$

and c_{13}^m and c_{33}^m are given by (4.3). One can also present the matrix $\mathbf{c}_m$ in the engineering notation, namely,

$$
\begin{bmatrix} c_{11}^m & c_{12}^m & c_{13}^m \\ c_{12}^m & c_{11}^m & c_{13}^m \\ c_{13}^m & c_{13}^m & c_{33}^m \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1^m} & -\dfrac{v_{12}^m}{E_1^m} & -\dfrac{v_{13}^m}{E_1^m} \\ -\dfrac{v_{21}^m}{E_1^m} & \dfrac{1}{E_1^m} & -\dfrac{v_{13}^m}{E_1^m} \\ -\dfrac{v_{31}^m}{E_3^m} & -\dfrac{v_{31}^m}{E_3^m} & \dfrac{1}{E_3^m} \end{bmatrix}^{-1}, \tag{4.5}
$$

see, for example, Christensen (1979). Here E_i^m is the Young's modulus in the ith direction and v_{ij}^m is a Poisson's ratio in the $\{ij\}$-plane. Note that the matrix on the right-hand side is symmetric, i.e., $v_{12}^m = v_{21}^m$ and $v_{13}^m E_3^m = v_{31}^m E_1^m$. Equation (4.5) leads to the relations

$$E_1 = \frac{4(1+x^2)\lambda_1 \lambda_2 \mu^m}{\lambda_1 \lambda_2 (1+x^2) + \mu^m (\lambda_1 x^2 + \lambda_2)}, \quad E_3 = \frac{\lambda_1 \lambda_2 (1+x^2)}{\lambda_1 + \lambda_2 x^2}, \tag{4.6}$$

$$v_{12} = v_{21} = \frac{\lambda_1 \lambda_2 (1+x^2) - \mu^m (\lambda_1 x^2 + \lambda_2)}{\lambda_1 \lambda_2 (1+x^2) + \mu^m (\lambda_1 x^2 + \lambda_2)}, \quad v_{31} = -\frac{(\lambda_1 - \lambda_2)x}{2(\lambda_1 + \lambda_2 x^2)}, \quad v_{13} = v_{31} \frac{E_1}{E_3}.$$

$$\tag{4.7}$$

These expressions will help us to interpret the results of optimization. We use the program Maple V (1981) to obtain these formulas and to solve the optimization problem.

5. NUMERICAL RESULTS AND DISCUSSION

Our numerical experiments (using MAPLE V program) show that the absolute value of $|d_h|$, and also $d_h g_h$ and k_h are decreasing functions of the parameters δ, μ^m, and λ_1. Therefore, these parameters were chosen to lie on their lower bounds defined by the condition $\delta \geqslant 0$ and by the restrictions (3.10) and (3.11), i.e.

$$\delta = 0, \quad \mu^m = a\mu^p, \quad \lambda_1 = a\mu^p. \tag{5.1}$$

This is important, because it defines the optimal shape and arrangement of the piezoelectric rods. Namely, $\delta = 0$ means that the optimal structures are Hashin (1965)

coated-cylinders assemblages with the bulk modulus K^- [see (3.2)]. Such micro-structures are good only for theoretical purposes because clearly it is impossible to prepare a composite which involves infinitely many length-scales, as is the case for the coated-cylinders assemblages. Different microstructures with the same bulk modu-lus were discovered by Vigdergauz (1989, 1994). He found the shape of the rods such that a composite consisting of a square array of these rods in a matrix will possess transverse bulk modulus K^-. Recently Vigdergauz (1996) generalized a result and found the shape of the rods such that a composite containing a hexagonal array of these rods will be transversely isotropic and will possess bulk modulus K^-. For low volume fraction the shapes of optimal rods are very close to circular cylinders. Therefore a transversely isotropic composite consisting of a hexagonal array of cir-cular rods in a polymer matrix is optimal for hydrophone designs.

We also found that $|d_h|$, $d_h g_h$ and k_h are increasing functions of λ_2. Therefore, the parameters λ_2, and x should be chosen so as to satisfy the restrictions (3.8) as an equality, i.e.

$$\det \begin{pmatrix} K^p - K^m & c_{12}^p - c_{13}^m \\ c_{12}^p - c_{13}^m & c_{11}^p - c_{33}^m \end{pmatrix} = 0. \qquad (5.2)$$

Here K^m, c_{13}^m, and c_{33}^m should be expressed in terms of $\lambda_1 = a\mu^p$, λ_2 and x [cf. (4.3) and (4.4)]. Equation (5.2) has the solution

$$\lambda_2 = \mu^p \frac{x^2(3K^p - aK^p - \mu^p) - 2ax(K^p - \mu^p) + K^p(3-a) - \mu^p(1+a)}{x^2(K^p + \mu^p(1-a)) - 2x(K - \mu^p) + K^p - a\mu^p}. \qquad (5.3)$$

In summary, we have specified all of the design variables except the volume fraction f of the piezoceramic rods and the parameter x. We will find these parameters numerically.

Figure 2 depicts the dependence of the function d_h on the control parameters x and f in the rectangular domain $x \in [-3, 3]$, $f \in [0, 1]$. Figure 3 depicts a "slice" from Fig. 2 at $f = 0.1$. These pictures are generic for all of the mentioned optimization problems. Note that positive values of x correspond to positive values of the Poisson's ratio

$$v_{31} = \frac{(\lambda_2 - \lambda_1)x}{2(\lambda_1 + \lambda_2 x^2)}, \qquad (5.4)$$

because $\lambda_2 \geqslant \lambda_1$.

As can be seen from Fig. 3 (which is generic for all the performance characteristics such as $|d_h|$, $d_h g_h$, and k_h), each of these functionals in the optimization problems (i)–(iii) has two maxima as functions of the parameters x at fixed f. One of these maxima corresponds to positive values of x, and the other to negative values of this parameter.

By using Maple V we found all of these maxima. The results are summarized in Table 1 where we compare the performance characteristics of pure PZT ceramic to three different groups of optimal design projects.

The first row in Table 1 corresponds to the values of the parameter for the pure PZT ceramic with the moduli given by (3.12). Group 2 (rows 2.1–2.4 of Table 1) corresponds to a "basic" optimal design of the hydrophone made of PZT and isotropic
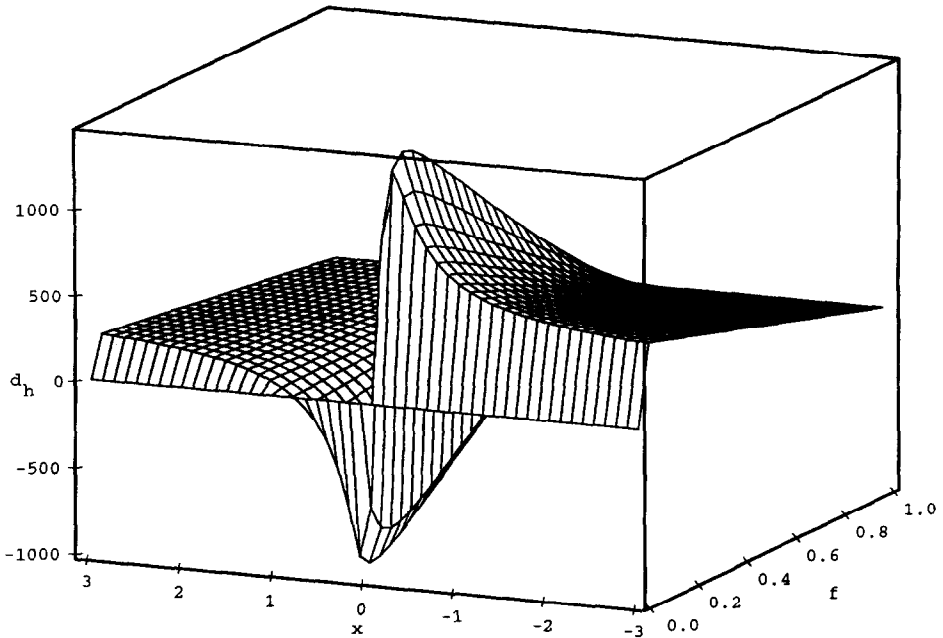
Fig. 2. Dependence of the hydrophone coupling factor d_h on the control parameter that defines the eigenvector directions, x [see (4.2)] and volume fraction of the PZT rods, f.
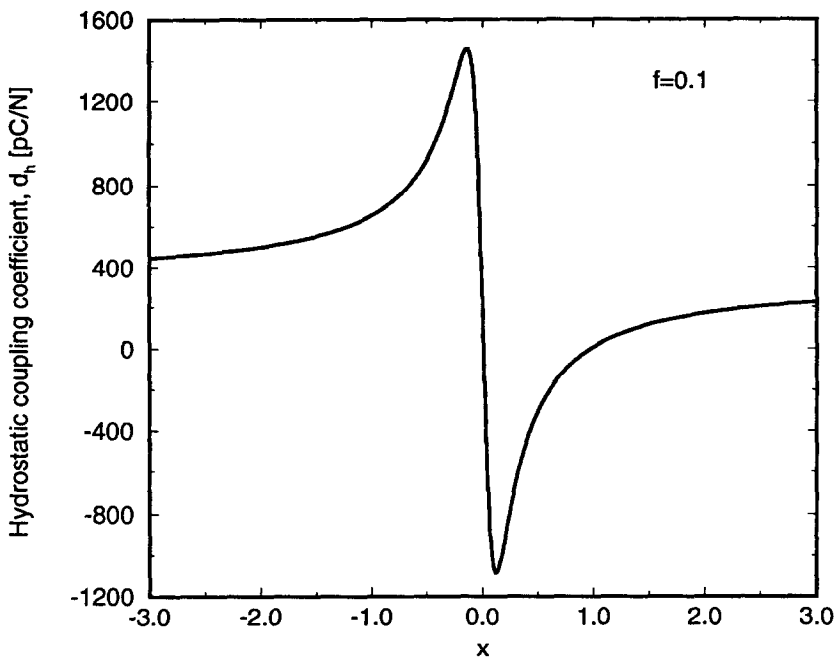

Fig. 3. Dependence of the hydrophone coupling factor d_h on the control parameter that defines the eigenvector directions, x at $f = 0.1$.

Table 1. *Hydrophone performance characteristics of pure PZT, piezocomposite with isotropic polyurethane matrix, and piezocomposites with optimal transversely isotropic matrices with either positive or negative Poisson's ratios* v_{31}. *Asterisk denotes the cost function(s) for the specific project*

N	project	f	x	v_{31}^m	d_h (pC/N)	$d_h g_h$ (μm^2/N)	k_h	k_h^2
1	PZT5A	1.000	N/A	N/A	32.0	0.068	0.078	0.0061
2	basic:							
2.1	best d_h	0.212	N/A	0.370	66.8*	1.496	0.082	0.0067
2.2	best $d_h g_h$	0.036	N/A	0.370	40.4	3.865*	0.116	0.0135
2.3	best k_h	0.041	N/A	0.370	43.3	3.848	0.116*	0.0135
2.4	good for all	0.076	N/A	0.370	56.2*	3.262*	0.110*	0.0121
3	$v_{31}^m \geqslant 0$:							
3.1	best d_h	0.109	0.124	2.513	−1087*	766	0.238	0.0566
3.2	best $d_h g_h$	0.014	0.170	2.259	−645	2200*	0.367	0.1347
3.3	best k_h	0.010	∞	0.000	352	700	0.556*	0.3091
3.4	good for all	0.028	0.210	2.005	−798*	1609*	0.407*	0.1656
4	$v_{31}^m \leqslant 0$:							
4.1	best d_h	0.098	−0.138	−2.107	1458*	1517	0.302	0.0912
4.2	best $d_h g_h$	0.006	−0.350	−1.240	762	5655*	0.458	0.2098
4.3	best k_h	0.017	−1.800	−0.270	537	1508	0.567*	0.3215
4.4	good for all	0.021	−0.200	−1.817	1147*	4445*	0.448*	0.2007

polyurethane without pores. The parameters to optimize are only the shape, cross-section, and volume fraction of PZT rods but not the elastic properties of the polymer matrix. These basic optimal design projects correspond closely to the ones studied by Avellaneda and Swart (1994). Row 2.1 of Table 1 describes the design that gives the maximal $|d_h|$, row 2.2 describes the project that optimizes $d_h g_h$, and row 2.3 optimizes k_h. Row 2.4 corresponds to a design that is good "on average", i.e. that has parameters $|d_h|$, $d_h g_h$, and k_h that are within 16% from their maxima in the rows 2.1–2.3.

Group 3 (rows 3.1–3.4 of Table 1) corresponds to optimal projects where we restrict ourselves to matrix materials with a positive Poisson's ratio $v_{31}^m \geqslant 0$ (i.e., $x \geqslant 0$). Good "on average" design (row 3.4) has parameters $|d_h|$, $d_h g_h$, and k_h equal to 73% of their maxima in the rows 3.1–3.3.

Group 4 (rows 4.1–4.4 of Table 1) corresponds to optimal projects where we do not assume that matrix Poisson's ratio v_{31}^m is positive. In fact, v_{31}^m is negative for all these examples. Good "on average" design (row 4.4) has parameters $|d_h|$, $d_h g_h$, and k_h equal to 78% of their maxima in the rows 4.1–4.3. These are more complicated structures but allow one to achieve higher hydrophone performance characteristics.

6. SENSITIVITY ANALYSIS

In this section we analyze dependence of the performance of optimal hydrophones on the design variables x, f, and on the parameter a.

All of the functionals that we have considered (i.e. $|d_h|$, $d_h g_h$ and k_h) strongly depend
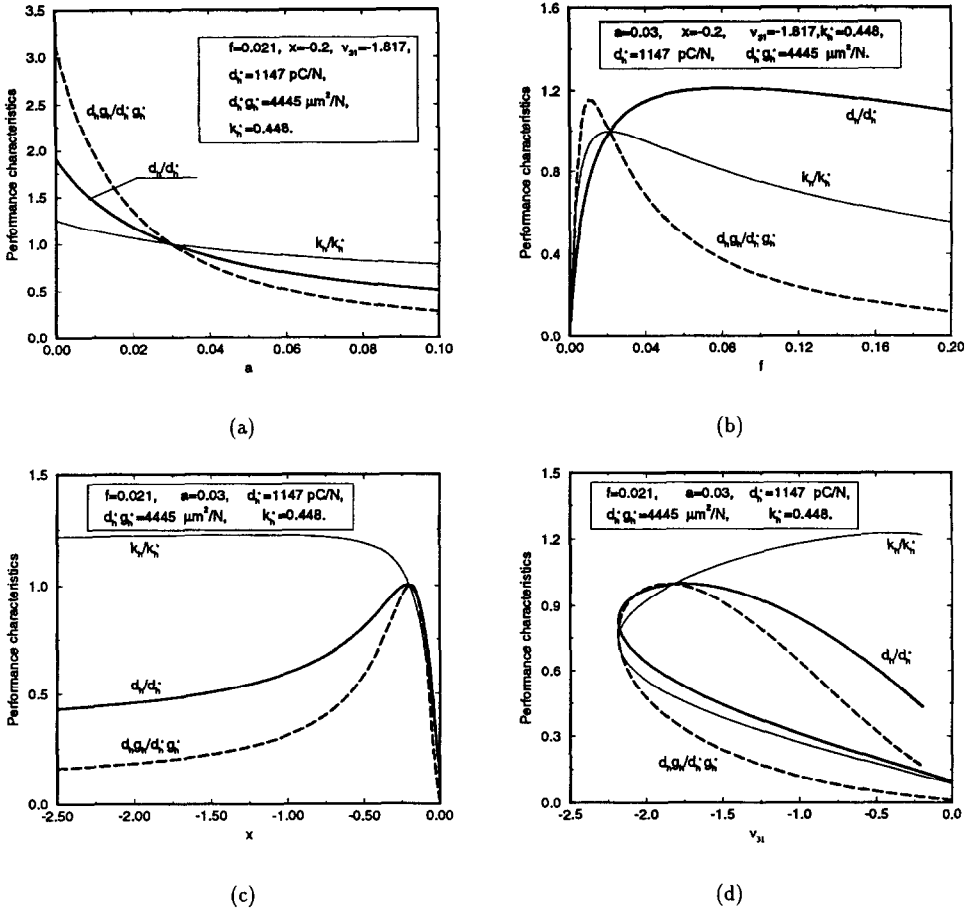
Fig. 4. Dependence of the scaled performance characteristics of the hydrophone on the various parameters for the project 4.4: (a) parameter that restricts the matrix properties, a [see (3.10) and (3.11)]; (b) volume fraction of the PZT rods, f; (c) parameter that defines the eigenvector directions, x [see (4.2)]; (d) Poisson's ratio of the matrix, v_{31}^m [see (4.6) and (4.7)]. See also Table 1.

on the parameter a that restricts from below the stiffness of the matrix material through the inequalities (3.10) and (3.11). Decreasing this parameter allows one to enhance the performance of the piezocomposite. As an example, consider project 4.4 of Table 1. Let us fix

$$f = 0.021, \quad x = -0.20, \quad v_{31}^m = -1.817 \tag{6.5}$$

and vary a. Figure 4(a) illustrates the dependence of the functions d_h/d_h^*, $d_h g_h/d_h^* g_h^*$, k_h/k_h^* on the parameter a in the interval $a \in [0.0, 0.1]$, where d_h^*, $d_h^* g_h^*$, k_h^* are the values from Table 1 (project 4.4). Optimizing over x and f for smaller a may give an even more substantial effect.

The subsequent companion figures illustrate the dependence of the ratios d_h/d_h^*, $d_h g_h/d_h^* g_h^*$, and k_h/k_h^*:

  (i) on the volume fraction f in the interval $f \in [0.0, 0.2]$ for $a = 0.03$ and $x = -0.20$
      [Fig. 4(b)],
 (ii) on the parameter x in the interval $x \in [-2.5, 0]$ for $a = 0.03$ and $f = 0.021$ [Fig.
      4(c)], and
(iii) on the Poisson's ratio v_{31}^m, for a fixed $a = 0.03$ and $f = 0.021$ [Fig. 4(d)].

One can see that it is important to keep the properties of the matrix and the volume
fraction of the rods in a small neighborhood of the optimal values in order to preserve
the optimality of the structure.

    These figures also illustrate the optimality requirement used to find the projects 3.4
and 4.4 that are "good on average". We tried to keep the ratios d_h/d_h^*, $d_h g_h/d_h^* g_h^*$,
k_h/k_h^* for these projects as close to one as possible. Then, the conditions

$$\frac{d_h}{d_h^*} = \frac{d_h g_h}{d_h^* g_h^*} = \frac{k_h}{k_h^*} \tag{6.6}$$

define the design variables x and f uniquely for a given value a.

7. OPTIMAL COMPOSITES

    In this section we study the design of optimal composites. Let us now turn our
attention to the projects 3.4 and 4.4 in Table 1. Project 3.4 is an optimal design made
of a matrix with positive Poisson's ratio v_{31}^m. The parameter $x = 0.21$ defines the
stiffness matrix $\mathbf{c}_m$ with coefficients

$$c_{11}^m = 0.2007, \quad c_{12}^m = 0.1460, \quad c_{13}^m = 0.6951, \quad c_{33}^m = 3.338, \tag{7.7}$$

or, in the equivalent form (4.5),

$$E_1^m = 0.0559, \quad E_3^m = 0.5501, \quad v_{12}^m = 0.021, \quad v_{13}^m = 0.204. \tag{7.8}$$

In this section all of the stiffness parameters (i.e. c_{ij}, E_i, etc.) are measured in GPa $= 10^9$
N/m². The upper three by three block of the stiffness tensor of such a material has
the following eigenvalues and eigenvectors:

$$\rho_1 = 3.632, \quad \mathbf{v}^{(1)} = (0.203 \ \ 0.203 \ \ 0.958),$$

$$\rho_2 = 0.0547, \quad \mathbf{v}^{(2)} = (-0.707 \ \ 0.707 \ \ 0.0),$$

$$\rho_3 = 0.0526, \quad \mathbf{v}^{(3)} = (0.677 \ \ 0.677 \ \ -0.286). \tag{7.9}$$

It is seen that one eigenvalue ($\rho_1 = 3.632$) is of the order of the stiffness of the
polyurethane ($c_{11}^p = 4.422$) but the other two eigenvalues are significantly lower. This
material is transversely isotropic; it can be easily deformed by shear $\mathbf{v}_2$ in the x_1–x_2
plane, and in $\mathbf{v}^{(3)}$ direction, but strongly resists deformation in the $\mathbf{v}^{(1)}$ direction. A
schematic version of this material can be obtained in the following way: imagine stiff
sticks aligned in two cones with the axis in the x_3 direction, and bound together in

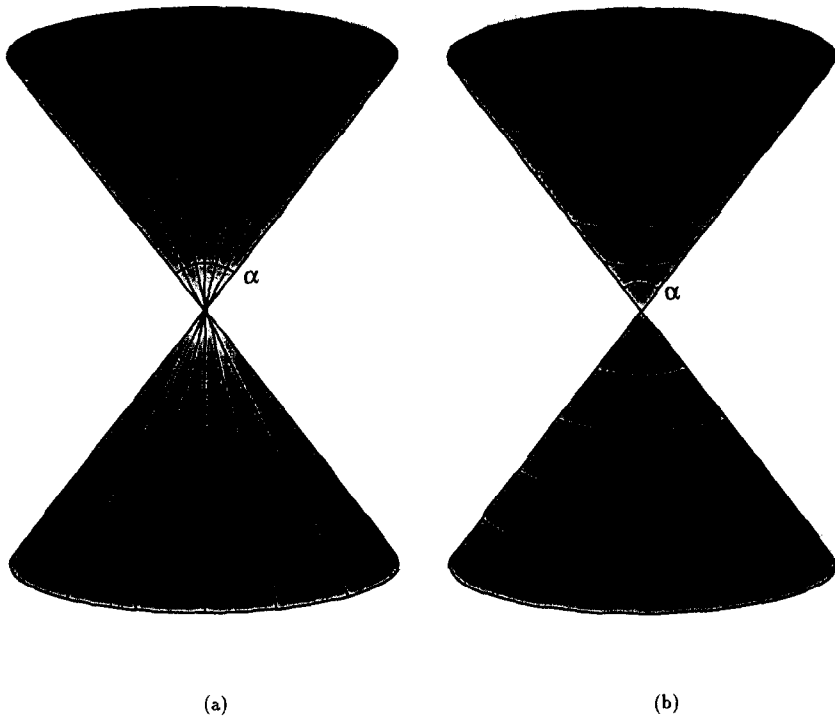(a)   (b)

Fig. 5. Schematic of the optimal microstructures for : (a) project 3.4 and (b) project 4.4.

the joint vertex of the cone [see Fig. 5(a)]. Such a cone is "transversely isotropic", it has "easy" modes of deformation $v^{(2)}$ and $v^{(3)}$ if the cone angle α is defined by the formula

$$\tan\frac{\alpha}{2} = -\frac{v_3^{(3)}}{\sqrt{2}v_1^{(3)}} = \frac{\sqrt{2}v_1^{(1)}}{v_3^{(1)}} = 0.299, \quad \alpha = 0.581, \quad (\alpha = 33°), \qquad (7.10)$$

and resists deformation $v^{(1)}$ (along the "sticks"). An alternative image that can be helpful is that of the cone made of sheets of metal that can slide, like in the baggage carousel in airports, along the marked lines [see Fig. 5(a)]. Such a cone will easily change its angle ($v^{(3)}$ deformation), and deform in the x_1–x_2 plane ($v^{(2)}$ deformation), but will resist to shrinking in the $v^{(1)}$ direction. Combining such cones in a material-like structure, one can achieve the desired stiffness tensor of the matrix.

For project 4.4, $x = -0.20$ and the components of the stiffness matrix c_m are given by

$$c_{11}^m = 0.1276, \quad c_{12}^m = 0.0729, \quad c_{13}^m = -0.3644, \quad c_{33}^m = 1.849, \qquad (7.11)$$

or, equivalently,

$$E_1^m = 0.0558, \quad E_3^m = 0.5248, \quad v_{12}^m = 0.019, \quad v_{13}^m = -0.193. \qquad (7.12)$$

The upper three by three block of this stiffness tensor has the following eigenvalues and eigenvectors

$$\rho_1 = 1.997, \quad \mathbf{v}^{(1)} = (0.195 \ 0.195 \ -0.961),$$

$$\rho_2 = 0.0547, \quad \mathbf{v}^{(2)} = (-0.707 \ 0.707 \ 0.0),$$

$$\rho_3 = 0.0526, \quad \mathbf{v}^{(3)} = (0.680 \ 0.680 \ 0.276). \tag{7.13}$$

Again, one of these eigenvalues ($\rho_1 = 1.997$) is of the order of the stiffness of the polyurethane, and the other two are significantly lower. This material is also transversely isotropic; it can be easily deformed by shear in the x_1–x_2 plane, and in $\mathbf{v}^{(3)}$ direction, but strongly resists deformation in the $\mathbf{v}^{(1)}$ direction. The schematic version of this material is illustrated by Fig. 5(b). Again, imagine two cones with a joint vortex, having an axis in the x_3-direction and with a vertex angle α defined by the formula

$$\tan\frac{\alpha}{2} = -\frac{v_3^{(1)}}{\sqrt{2}v_1^{(1)}} = \frac{\sqrt{2}v_1^{(3)}}{v_3^{(3)}} = 3.48, \quad \alpha = 2.58, \quad (\alpha = 148°). \tag{7.14}$$

Let such a cone have the following special properties. It resists deformation in the direction $\mathbf{v}^{(1)}$ (i.e. to changing the angle of the cone), like a cone made of a thin metal. It does not resist deformation in the direction $\mathbf{v}^{(1)}$ (i.e. to changing the shape of the x_1–x_2 plane cross-section of the cone from circle to ellipses), again like a cone made of a thin metal. However, it has an unusual feature: it does not resist deformation along the base of the cone, as if the "sticks" in Fig. 5(a) would resist bending but would not resist stretching. Thus, it behaves like a "telescopic cone", with sliding surfaces situated as marked in Fig. 5(b), corresponding to negative values of the Poisson's ratio $v_{31}^m = -1.817$. Again, combining such imaginary cones in a material-like structure, one can achieve the desired stiffness tensor.

We emphasize that Figs 5(a) and (b) are only schematic constructions of the optimal matrix materials. The optimal microstructures may have nothing in common with our cone-like constructions, although they should possess the effective properties specified by our calculations. In contrast to the study of Sigmund et al. (1996), our aim was not to find real microstructures.

Let us now turn our attention to project 3.3. This has an interesting feature in that the optimal value of the parameter x is equal to infinity, and the optimal value of the Poisson's ratios $v_{13}^m = v_{31}^m = 0$ are equal to zero. The stiffness matrix $\mathbf{c}_m$ for this project has the coefficients

$$c_{11}^m = 2.002, \quad c_{12}^m = 1.947, \quad c_{13}^m = 0.0, \quad c_{33}^m = 0.02737, \tag{7.15}$$

and the following eigenvalues and eigenvectors of the upper three by three block of the stiffness matrix:

$$\rho_1 = 3.949, \quad \mathbf{v}^{(1)} = (0.707 \ 0.707 \ 0),$$

$$\rho_2 = 0.0547, \quad \mathbf{v}^{(2)} = (-0.707 \ 0.707 \ 0),$$

$$\rho_3 = 0.0274, \quad \mathbf{v}^{(3)} = (0 \ 0 \ 1). \tag{7.16}$$

As can be seen, such a material resists only compression in the x_1–x_2 plane. It can be approximately modeled by a laminate material with a polymer and void layers alter-

native along the x_3-direction. In order to ensure small stiffness in the x_3-directions, one needs to connect the polymer layers by small polymer rods. Numerical experiments show that the performance of the piezocomposite with a matrix made of such a laminate composite almost matches the performance of the piezocomposite with the optimal matrix (7.15).

8. SUMMARY OF THE RESULTS

We have investigated analytically and numerically the optimal design problems of a piezoelectric hydrophone made of PZT ceramic and polymer matrix weakened by an optimal arrangement of the pores. We have assumed that the matrix material, piezoceramic rods, and the composite are transversely isotropic. Our investigation allows us to formulate the following results:

- Optimal piezocomposites consist of hexagonal arrays of piezoelectric rods in a soft anisotropic polymer matrix.
- The volume fraction of the piezoelectric rods should be small; it varies in the interval $f \in [0.01, 0.11]$ depending on the particular functional (see Table 1).
- The optimal matrix material is highly anisotropic, with one eigenvalue of the stiffness matrix being as large as possible, and the others being as small as allowed by the design restrictions. Moreover, matrix materials with negative Poisson's ratios v_{31}^m $\leqslant 0$ deliver better results.
- The optimal design is very sensitive to the variation of the volume fraction of the rods and properties of the polymer matrix.
- For the chosen values of the parameters, maximal enhancement of the hydrophone characteristics (compared with the values for the pure PZT) is approximately 45 for the hydrostatic coupling coefficient d_h, 83,000 for the hydrophone figure of merit $d_h g_h$, and 7.27 for the electromechanical coupling factor k_h. Compared with values for the composite of isotropic polymer matrix and PZT, the optimal composites with anisotropic polymer matrix give maximal enhancement d_h by a factor of 22, $d_h g_h$ by a factor of 1460, and k_h by a factor of 4.89, respectively (see Table 1).
- Our results give theoretical upper bounds on the hydrophone characteristics, i.e. the actual performance will be lower than our estimates. Nevertheless, we consider these bounds to be very helpful, because they allow one to estimate the possible enhancement of the hydrophone performance due to structural optimization.
- Although we have not obtained the specific details of the microstructures, our results provide a helpful guideline as to the basic features of the effective properties of optimal piezocomposites, such as extreme anisotropy and the optimal direction of the "stiff" mode of the elasticity tensor of the optimal matrix material.

ACKNOWLEDGEMENTS

REFERENCES

Avellaneda, M. and Swart, P. J. (1994) Calculating the performance of 1-3 piezocomposites for hydrophone applications : an effective medium approach. Working paper, Courant Institute of Mathematical Sciences.

Chan, H. W. W. and Unsworth, J. (1989) Simple model for piezoelectric ceramic/polymer 1-3 composites used in ultrasonic transducer applications. *IEEE Tran. Ultrason. Ferro. Freq. Control* **36**, 434–441.

Christensen, R. M. (1979) *Mechanics of Composite Materials.* Wiley–Interscience, NY.

Hashin, Z. (1965) On elastic behavior of fibre reinforced materials of arbitrary transverse phase geometry. *J. Mech. Phys. Solid* **13**, 119–134.

Haun, M. J. and Newnham, R. E. (1986) An experimental and theoretical study of 1-3 and 1-3-0 piezoelectric PZT-polymer composites for hydrophone applications. *Ferroelec.* **68**, 123–139.

Klicker, K. A., Biggers, J. V. and Newnham, R. E. (1981) Composites of PZT and epoxy for hydrostatic transducer applications. *J. Am. Ceram. Soc.* **64**, 5–9.

Lakes, R. (1987) Foam structures with a negative Poisson's ratio. *Science* **235**, 1038–1040.

MAPLE V, Release 2, Princeton University, Copyright© 1981–1992 by the University of Waterloo.

Milton, G. M. and Cherkaev, A. V. (1995a) Which elasticity tensors are realizable? *J. Engng Mater. Tech.* **117**, 483–493.

Milton, G. M. and Cherkaev, A. V. (1995b) private communication.

Newnham, R. E. (1986) Composite electroceramics. *Ferroelec.* **68**, 1–32.

Newnham, R. E. and Ruschau, G. R. (1991) Smart electroceramics. *J. Am. Ceram. Soc.* **74**, 463–480.

Sigmund, O., Torquato, S. and Aksay, I. A. (1996) On the design of 1-3 piezocomposites using topology optimization, in preparation.

Smith, W. A. (1991) Optimizing electromechanical coupling in piezocomposites using polymers with negative Poisson's ratio. *Proc.* 1991 *IEEE Ultrason. Symp.*, 661–666.

Smith, W. A. (1993) Modeling 1-3 composite piezoelectrics : hydrostatic response. *IEEE Tran. Ultrason. Ferro. Freq. Control* **40**, 41–49.

Ting, R., Y., Shalov, A. A. and Smith, W. A. (1990) Piezoelectric properties of 1-3 composites in calcium-modified lead titanate in epoxy resins. *Proc.* 1990 *IEEE Ultrason. Symp.*, 707–710.

Vigdergauz, S. B. (1989) Regular structures with extremal elastic properties. *Mech. Solids* **24**, 57–63.

Vigdergauz, S. B. (1994). Two-dimensional grained composites of extreme rigidity. *J. Appl. Mech.* **3**, 390–394.

Vigdergauz, S. B. (1996) private communication.