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PHASE-INTERCHANGE RELATIONS FOR THE ELASTIC MODULI OF TWO-PHASE COMPOSITES

L. V. GIBIANSKY and S. TORQUATO

Department of Civil Engineering and Operations Research, and Princeton Materials Institute, Princeton University, Princeton, NJ 08544 U.S.A.

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Abstract—For isotropic two-phase composites, we derive phase-interchange inequalities for the bulk and shear moduli in two and three space dimensions. We find optimal microstructures that realize part of the bulk and shear moduli bounds in two dimensions. Geometrical-parameter bounds and the translation method are used to prove the results. The phase-interchange relations are applied to composites with cavities or a perfectly rigid phase, composites near the percolation threshold, incompressible composites, composites with equal bulk or shear phase moduli, and symmetric composites. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

This paper is concerned with phase-interchange relations for elastic two-phase composites. The basic question can be phrased as follows: given the effective moduli of a particular composite having two phases, what are the effective moduli of the same composite but with the phases interchanged? Such relations are well-known for the conductivity problem. Keller [1], Mendelson [2] and Dykhne [3] independently proved that in two-dimensional space, the effective conductivity $\sigma_*(\sigma_1, \sigma_2)$ of a two-phase isotropic composite with phase conductivities σ_1 and σ_2 is related to the effective conductivity $\sigma_*(\sigma_2, \sigma_1)$ of the "phase-interchanged" composite by the equality

$$\sigma_*(\sigma_1, \sigma_2)\sigma_*(\sigma_2, \sigma_1) = \sigma_1\sigma_2. \tag{1.1}$$

Here the composites with effective conductivity $\sigma_*(\sigma_1, \sigma_2)$ and $\sigma_*(\sigma_2, \sigma_1)$ have local conductivities given by

$$\sigma(x) = \chi(x)\sigma_1 + [1 - \chi(x)]\sigma_2, \tag{1.2}$$

and

$$\hat{\sigma}(x) = \chi(x)\sigma_2 + [1 - \chi(x)]\sigma_1, \tag{1.3}$$

respectively, where $\chi(x)$ is the characteristic function of the microstructure (i.e. $\chi(x) = 1$ in one phase and zero otherwise).

One can use equality (1.1) to test bounds on the effective conductivity. Milton [4] employed it to prove some important analytical properties of the function $\sigma_*(\sigma_1, \sigma_2)$ that lead to restrictive bounds on this function. Moreover, by using (1.1) or similar relations in three dimensions [see equations (1.5) and (1.6) below], Milton [4] obtained cross-property bounds on the effective conductivity $\sigma_*(\sigma_1, \sigma_2)$ of the composite in terms of the conductivity $\sigma_*(\sigma_1^+, \sigma_2^+)$ of the composite with the same microstructure, but with different phase conductivities. Note that relation (1.1) is valid even for materials with complex conductivity.

If the composite is symmetric in the sense that $\sigma_*(\sigma_1, \sigma_2) = \sigma_*(\sigma_2, \sigma_1)$, then

$$\sigma_*(\sigma_1, \sigma_2) = \sigma_*(\sigma_2, \sigma_1) = \sqrt{\sigma_1\sigma_2}, \tag{1.4}$$

as follows from (1.1). The checkerboard is an obvious example of a *symmetric* material and has an effective conductivity given by (1.4). The phase-interchange equality (1.1) was generalized for an anisotropic composite by Mendelson [2]. Cherkaev and Gibiansky [5] also obtained an anisotropic version of the equality (1.1) as a corollary of more general cross-property bounds on the electrical and magnetic properties of two-dimensional anisotropic composites.

In three-dimensional space, the phase-interchange equality in the form of (1.1) is not valid. It is replaced by the system of inequalities given by

$$\frac{f_1 f_2}{f_1\sigma_1 + f_2\sigma_2 - \sigma_*} + \frac{f_1 f_2}{f_1\sigma_2 + f_2\sigma_1 - \hat{\sigma}_*} \leqslant \frac{3(\sigma_1+\sigma_2)}{(\sigma_1-\sigma_2)^2}, \tag{1.5}$$

$$\frac{\sigma_*\hat{\sigma}_*}{\sigma_1\sigma_2} + \frac{\sigma_* + \hat{\sigma}_*}{\sigma_1+\sigma_2} \geqslant 2, \tag{1.6}$$

where f_1 and $f_2 = 1 - f_1$ are the volume fractions of the phases. Here and henceforth, the phase-interchanged effective conductivity is denoted by $\hat{\sigma}_* = \sigma_*(\sigma_1, \sigma_1)$. Relation (1.5) is an obvious corollary of the Bergman's bound [6]. Relation (1.6) was conjectured by Milton [4] and later proved by Avellaneda *et al.* [7]; it is always sharper than the bound $\sigma_*\hat{\sigma}_* \geqslant \sigma_1\sigma_2$ obtained by Schulgasser [8].

The phase-interchange inequality (1.6) has been generalized for a multiphase composite by Nesi [9]. Bruno [10] has introduced and exploited the notion of infinite phase-interchangeability to study the perturbation expansion for the effective conductivity of symmetric materials.

The main focus of this paper is to find phase-interchange relations for the elastic moduli of composites. We consider an isotropic composite that is built from two linearly elastic isotropic materials, taken in prescribed proportions f_1 and f_2, and the corresponding phase-interchanged composite. We denote by κ_1, κ_2 and μ_1, μ_2 the bulk and shear moduli of the first and second phases, respectively. We also denote by κ_*, $\hat{\kappa}_*$ and μ_*, $\hat{\mu}_*$ the effective bulk and shear moduli of the original composite and the phase-interchanged composite. We seek to find relations between the bulk moduli κ_* and $\hat{\kappa}_*$ and between the shear moduli μ_* and $\hat{\mu}_*$.

Only a few phase-interchange relations exist for the elastic moduli, and only in two dimensions. The phase interchange equality similar to (1.1) was found by Berdichevsky [11], who proved that for a two-dimensional, two-phase composite with *incompressible phases* the following equality holds:

$$\mu_*\hat{\mu}_* = \mu_1\mu_2. \tag{1.7}$$

For a two-dimensional composite consisting of two *phases with equal bulk moduli,* this relation was generalized by Helsing *et al.* [12] as follows:

$$E_*\hat{E}_* = E_1 E_2. \tag{1.8}$$

Here $E_1 = 4\kappa\mu_1/(\kappa+\mu_1)$, $E_2 = 4\kappa\mu_2/(\kappa+\mu_2)$ are the Young's moduli of the phases, and $E_* = 4\kappa\mu_*/(\kappa+\mu_*)$, $\hat{E}_* = 4\kappa\hat{\mu}_*/(\kappa+\hat{\mu}_*)$ are the Young's moduli of the composite and phase-interchanged composite, respectively. We will show that these results follow directly from our general bounds for arbitrary values of the phase moduli. Moreover, we shall obtain phase-interchange relations for a three-dimensional composite.

We use two methods to prove the bounds, namely, the Beran-type, geometrical-parameter bounds and the so-called translation method. One can find a detailed review of geometrical-parameter bounds in our recent paper [13]. There we summarized the existing geometrical-parameter bounds and have improved the bounds for the plane elasticity problem using the translation method. The translation method was introduced by Cherkaev and Lurie [14, 15] and by Murat and Tartar [16, 17]. The procedure that we use here has been described in a number of papers [18–21]. Accordingly, we will omit details of the translation method.

In general, the translation method delivers better results, but is more involved than the procedure that relies on the geometrical-parameter bounds. Therefore, we prefer to use geometrical-parameter bounds and use the translation method only when it allows us to improve upon the geometrical-parameter bounds.

Let us introduce some helpful notation. First, let F be a function of five variables

$$F(\kappa_1, \kappa_2, f_1, f_2, y) = f_1\kappa_1 + f_2\kappa_2 - \frac{f_1 f_2(\kappa_1 - \kappa_2)^2}{f_2\kappa_1 + f_1\kappa_2 + y}, \tag{1.9}$$

which will be used to simplify our expressions. Let us also introduce a function

$$y(\kappa_1, \kappa_2, f_1, f_2, \kappa_*) = -f_1\kappa_2 - f_2\kappa_1 + \frac{f_1 f_2(\kappa_1 - \kappa_2)^2}{f_1\kappa_1 + f_2\kappa_2 - \kappa_*}. \tag{1.10}$$

This function is an inverse function to F (of its fifth variable). It is a scalar form of the fractional linear Y-transformation that used to simplify the proof of the bounds and to present the results in a convenient form (see Milton [22]; Cherkaev and Gibiansky [5]). Bounds in terms of the function y (which we will refer to as Y-transformation) are usually much simpler than those in the original form. For the sake of brevity, we will omit the first four arguments of the functions F and y and will use the notation

$$F_\kappa(y_\kappa) = F(\kappa_1, \kappa_2, f_1, f_2, y_\kappa), \qquad F_\mu(y_\mu) = F(\mu_1, \mu_2, f_1, f_2, y_\mu), \tag{1.11}$$

$$y_\kappa(\kappa_*) = y(\kappa_1, \kappa_2, f_1, f_2, \kappa_*), \qquad y_\mu(\mu_*) = y(\mu_1, \mu_2, f_1, f_2, \mu_*). \tag{1.12}$$

Similarly, for the phase-interchanged composite, one can introduce the functions $\hat{F}$ and $\hat{y}$

$$\hat{F}(\kappa_1, \kappa_2, f_1, f_2, y) = f_2\kappa_1 + f_1\kappa_2 - \frac{f_1 f_2(\kappa_1 - \kappa_2)^2}{f_1\kappa_1 + f_2\kappa_2 + y},$$

$$\hat{y}(\kappa_1, \kappa_2, f_1, f_2, \hat{\kappa}_*) = -f_1\kappa_1 - f_2\kappa_2 + \frac{f_1 f_2(\kappa_1 - \kappa_2)^2}{f_1\kappa_2 + f_2\kappa_1 - \hat{\kappa}_*}. \tag{1.13}$$

Note that

$$\hat{F}(\kappa_1, \kappa_2, f_1, f_2, y) = F(\kappa_2, \kappa_1, f_1, f_2, y) = F(\kappa_1, \kappa_2, f_2, f_1, y),$$

$$\hat{y}(\kappa_1, \kappa_2, f_1, f_2, \kappa_*) = y(\kappa_1, \kappa_1, f_1, f_2, \kappa_*) = y(\kappa_1, \kappa_2, f_2, f_1, \kappa_*). \tag{1.14}$$

We will use the notation

$$\hat{F}_\kappa(y_\kappa) = \hat{F}(\kappa_1, \kappa_2, f_1, f_2, y_\kappa), \qquad \hat{F}_\mu(y_\mu) = \hat{F}(\mu_1, \mu_2, f_1, f_2, y_\mu), \tag{1.15}$$

and

$$\hat{y}_\kappa(\hat{\kappa}_*) = \hat{y}(\kappa_1, \kappa_2, f_1, f_2, \hat{\kappa}_*), \qquad \hat{y}_\mu(\hat{\mu}_*) = \hat{y}(\mu_1, \mu_2, f_1, f_2, \hat{\mu}_*). \tag{1.16}$$

The rest of the paper is organized as follows: in Section 2, we state, prove and discuss new phase-interchange inequalities for the bulk and shear moduli of macroscopically isotropic two-dimensional elastic composites consisting of two isotropic phases. In Section 3, we describe corresponding findings for three-dimensional composites. In Sections 4 and 5 we apply our bounds to composites with cavities or a perfectly rigid phase, composites near the percolation threshold, incompressible composites, and composites with equal bulk or shear phase moduli.

We also apply our phase-interchange relations to "symmetric composites". In a number of cases, the bounds that we found are sharp.

2. PHASE-INTERCHANGE RELATIONS FOR TWO-DIMENSIONAL COMPOSITES

All of the results given here apply to macroscopically isotropic, two-dimensional composites consisting of two isotropic phases.

2.1 Results for the bulk modulus

The following statement specifies phase-interchange relations for the bulk modulus in two dimensions:

Statement 1. The bulk moduli κ_* and $\hat{\kappa}_*$ of a two-dimensional composite are restricted by the inequalities

$$\frac{f_1 f_2 \kappa_*}{\kappa_*(f_1\kappa_2+f_2\kappa_1)-\kappa_1\kappa_2}+\frac{f_1 f_2 \hat{\kappa}_*}{\hat{\kappa}_*(f_1\kappa_1+f_2\kappa_2)-\kappa_1\kappa_2}\leqslant\frac{\kappa_1\kappa_2(\mu_1+\mu_2)+\mu_1\mu_2(\kappa_1+\kappa_2)}{\mu_1\mu_2(\kappa_1-\kappa_2)^2}, \tag{2.1}$$

$$\frac{\kappa_*\hat{\kappa}_*-\mu_1\mu_2}{\kappa_*\hat{\kappa}_*(\mu_1+\mu_2)+\mu_1\mu_2(\kappa_*+\hat{\kappa}_*)}\leqslant\frac{\kappa_1\kappa_2-\mu_1\mu_2}{\kappa_1\kappa_2(\mu_1+\mu_2)+\mu_1\mu_2(\kappa_1+\kappa_2)}. \tag{2.2}$$

In the κ_*–$\hat{\kappa}_*$ plane, (2.1) is a lower bound and (2.2) is an upper bound.

Proof of the lower bound (2.1). In Ref. [13] we have reformulated known bulk modulus geometrical-parameter bounds by Milton [23, 24] in terms of the Y-transformation (1.12) of the effective bulk modulus and have proved a new upper bound in two dimensions. Specifically, for the bulk modulus in two dimensions we have found that

$$\left[\frac{\zeta_1}{\mu_1}+\frac{\zeta_2}{\mu_2}\right]^{-1}\leqslant y_\kappa(\kappa_*)\leqslant F(\mu_1,\mu_2,\zeta_1,\zeta_2,\kappa_+), \tag{2.3}$$

where ζ_1 and $\zeta_2=1-\zeta_1$ are three-point geometrical parameters characterizing the structure, independent of the phases properties, and κ_+ is the maximal bulk modulus of the phases.

The same bounds for the phase-interchanged composite can be found as

$$\left[\frac{\zeta_1}{\mu_2}+\frac{\zeta_2}{\mu_1}\right]^{-1}\leqslant \hat{y}_\kappa(\hat{\kappa}_*)\leqslant \hat{F}(\mu_1,\mu_2,\zeta_1,\zeta_2,\kappa_+). \tag{2.4}$$

The lower bounds in (2.3) and (2.4) can be rewritten in the form

$$\frac{1}{y_\kappa(\kappa_*)}\leqslant\frac{\zeta_1}{\mu_1}+\frac{\zeta_2}{\mu_2},\qquad \frac{1}{\hat{y}_\kappa(\hat{\kappa}_*)}\leqslant\frac{\zeta_2}{\mu_1}+\frac{\zeta_1}{\mu_2}, \tag{2.5}$$

that immediately leads to the bound

$$\frac{1}{y_\kappa(\kappa_*)}+\frac{1}{\hat{y}_\kappa(\hat{\kappa}_*)}\leqslant\frac{1}{\mu_1}+\frac{1}{\mu_2}, \tag{2.6}$$

which is equivalent to (2.1).

The upper bound on the bulk modulus that can be obtained using (2.3) and (2.4) is weaker than the corresponding translation bound that we will obtain below.

Proof of the upper bound (2.2). As was done in Refs [18] and [20], we consider the functional that is the sum of the energies stored by the composite and the phase-interchanged

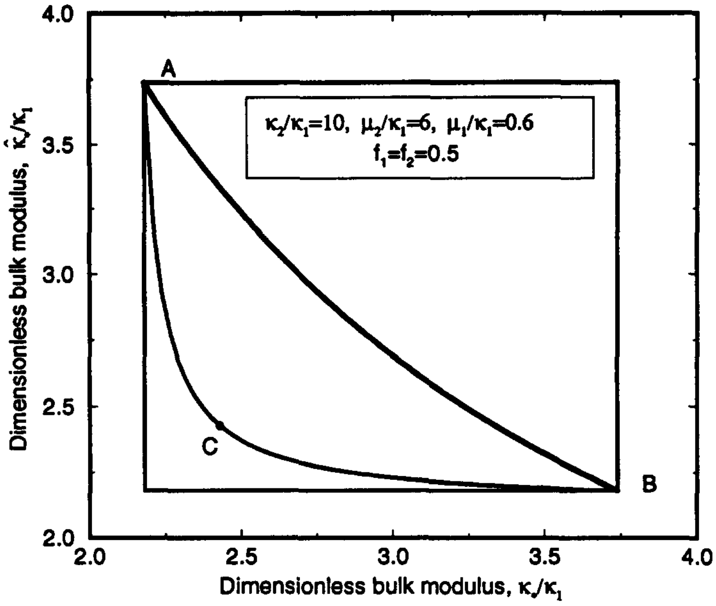

Fig. 1. Bulk moduli bounds in the κ_*–$\hat{\kappa}_*$ plane for a two-dimensional composite. The rectangle corresponds to the Hashin–Shtrikman bounds (2.11). The bold curve (upper bound (2.2)) corresponds to assemblages of doubly-coated circles described in the text. The points A and B correspond to the assemblages of coated circles. Point C corresponds to a special polycrystal described in the text.

composite in the stress fields with a given average. The translation bound on this sum leads to the inequality

$$(\kappa_*^{-1}+t_1)(\hat{\kappa}_*^{-1}+t_1)-t_2^2\geq 0, \tag{2.7}$$

which is valid for any t_1 and t_2 that satisfy to the matrix inequality

$$\begin{pmatrix} \kappa_1^{-1}+t_1 & 0 & t_2 & 0 \\ 0 & \mu_1^{-1}-t_1 & 0 & -t_2 \\ t_2 & 0 & \kappa_2^{-1}+t_1 & 0 \\ 0 & -t_2 & 0 & \mu_2^{-1}-t_1 \end{pmatrix}\geq 0. \tag{2.8}$$

By choosing

$$t_1=\frac{\mu_1^{-1}\mu_2^{-1}-\kappa_1^{-1}\kappa_2^{-1}}{\kappa_1^{-1}+\mu_1^{-1}+\kappa_2^{-1}+\mu_2^{-1}},\qquad t_2=\frac{\sqrt{(\kappa_1^{-1}+\mu_1^{-1})(\kappa_1^{-1}+\mu_2^{-1})(\kappa_2^{-1}+\mu_1^{-1})(\kappa_2^{-1}+\mu_2^{-1})}}{\kappa_1^{-1}+\mu_1^{-1}+\kappa_2^{-1}+\mu_2^{-1}} \tag{2.9}$$

to optimize the bound in (2.7) we arrive at the inequality (2.2) of Statement 1.

The results are depicted graphically in Fig. 1 for the following choice of the parameters:

$$\kappa_1=1,\qquad \mu_1=0.6,\qquad \kappa_2=10,\qquad \mu_2=6,\qquad f_1=f_2=0.5. \tag{2.10}$$

The rectangle in Fig. 1 corresponds to the Hashin–Shtrikman [25] bulk modulus bounds

$$\kappa_{1*}\leq\kappa_*\leq\kappa_{2*},\qquad \hat{\kappa}_{2*}\leq\hat{\kappa}_*\leq\hat{\kappa}_{1*}, \tag{2.11}$$

where

$$\kappa_{1*}=F_\kappa(\mu_1),\qquad \kappa_{2*}=F_\kappa(\mu_2), \tag{2.12}$$

$$\hat{\kappa}_{1*}=\hat{F}_\kappa(\mu_2),\qquad \hat{\kappa}_{2*}=\hat{F}_\kappa(\mu_1). \tag{2.13}$$

Bounds (2.1) and (2.2) enclose the lens-shaped region in the κ_*–$\hat{\kappa}_*$ plane that contains all the possible pairs $(\kappa_*, \hat{\kappa}_*)$. They intersect at the points $A=(\kappa_{1*}, \hat{\kappa}_{1*})$, $B=(\kappa_{2*}, \hat{\kappa}_{2*})$, that correspond to the Hashin [25] microstructures, i.e. to assemblages of coated circles that fill the whole space. the point $A = (\kappa_{1*}, \hat{\kappa}_{1*})$ corresponds to the assemblage where the second phase makes up the included circular core and the first phase forms the external coating. The point $B = (\kappa_{2*}, \hat{\kappa}_{2*})$ corresponds to the assemblage where the first phase makes up the core and the second phase forms the coating.

There exists one point $C = (\kappa_*^{(1)}, \hat{\kappa}_*^{(1)})$,

$$\kappa_*^{(1)} = F_\kappa\left(\left[\frac{f_1}{\mu_2} + \frac{f_2}{\mu_1}\right]^{-1}\right), \qquad \hat{\kappa}_*^{(1)} = \hat{F}_\kappa\left(\left[\frac{f_1}{\mu_1} + \frac{f_2}{\mu_2}\right]^{-1}\right), \tag{2.14}$$

on the lower bound (2.1) that corresponds to a polycrystal made of laminates of two phases by the procedure that was suggested by Avellaneda and Milton [26] and Rudelson [27]. The bulk modulus of such a composite has been found by Gibiansky and Milton [19].

The upper bound (2.2) (the bold curve in Fig. 1) is optimal since it corresponds to the assemblages of space-filling, doubly-coated circles. Each inclusion in these assemblages has a core made of one of the phases, coated with a circular layer of the other phase, and an external coating of the core phase. Even when the volume fractions of the phases are given, such microstructures possess one free parameter that controls the distribution of the core-phase material between the core and the outermost coating. By changing this parameter, one can obtain composites that correspond to any given point on the upper boundary (2.2). Such structures were suggested by Milton [4]. The expressions for their bulk moduli have been given by Gibiansky and Milton [19].

Interestingly, the region restricted by our inequalities has two narrow corners with the corner points $A = (\kappa_{1*}, \hat{\kappa}_{1*})$ and $B = (\kappa_{2*}, \hat{\kappa}_{2*})$. *Therefore, one can predict the effective bulk modulus $\hat{\kappa}_*$ of a phase-interchanged composite very precisely if the bulk modulus κ_* lies close to any of the Hashin–Shtrikman bounds κ_{1*} or κ_{2*} and thus to the Hashin–Shtrikman coated-circles assemblages.* Note also that the lower bound (2.1) depends on the phase volume fractions, whereas the upper bound (2.2) is independent of the volume fraction.

2.2 Results for the shear modulus

The phase-interchange relations for the shear moduli in two dimensions are summarized by the following statement:

Statement 2.

(i) The shear modulus upper bound in the μ_*–$\hat{\mu}_*$ plane is given by the inequality

$$\frac{\mu_1\mu_2 - \mu_*\hat{\mu}_*}{\mu_1 + \mu_2 - \mu_* - \hat{\mu}_*} \leqslant \frac{\mu_1\mu_2(\mu_+\kappa_- + \mu_-\kappa_+ + 2\mu_1\mu_2)}{\kappa_1\kappa_2(\mu_1+\mu_2) + \kappa_-\mu_+^2 + \kappa_+\mu_-^2 + 2\mu_1\mu_2(\mu_1+\mu_2+\kappa_1+\kappa_2)}. \tag{2.15}$$

Here and henceforth κ_- and μ_- (κ_+ and μ_+) are the minimal (maximal) bulk and shear moduli, respectively.

(ii) The shear modulus lower bound in the μ_*–$\hat{\mu}_*$ plane is given by the sharpest of the inequalities

$$\frac{\mu_1\mu_2 - \mu_*\hat{\mu}_*}{\mu_1 + \mu_2 - \mu_* - \hat{\mu}_*} \geqslant \frac{\mu_1\mu_2(\mu_+\kappa_+ + \mu_-\kappa_- + 2\mu_1\mu_2)}{\kappa_1\kappa_2(\mu_1+\mu_2) + \kappa_+\mu_+^2 + \kappa_-\mu_-^2 + 2\mu_1\mu_2(\mu_2+\mu_2+\kappa_1+\kappa_2)}, \tag{2.16}$$

$$\frac{f_1f_2\mu_*}{\mu_*(f_1\mu_2+f_2\mu_1) - \mu_1\mu_2} + \frac{f_1f_2\hat{\mu}_*}{\hat{\mu}_*(f_1\mu_1+f_2\mu_2) - \mu_1\mu_2} \leqslant \frac{2\kappa_1\kappa_2(\mu_1+\mu_2) + 2\mu_1\mu_2(\kappa_1+\kappa_2)}{\kappa_1\kappa_2(\mu_1-\mu_2)^2}. \tag{2.17}$$

Proof of the lower bound (2.17). According to Ref. [13] the Silnutzer [28] geometrical-parameter lower bound on the shear modulus of a two-dimensional composite can be presented in the form

$$\frac{1}{y_\mu(\mu_*)} \leqslant \frac{2\zeta_1}{\kappa_1} + \frac{2\zeta_2}{\kappa_2} + \frac{\eta_1}{\mu_1} + \frac{\eta_2}{\mu_2}. \tag{2.18}$$

Here $\eta_1 = 1 - \eta_2$ is yet another geometrical parameter of the structure. Similarly, for the phase-interchanged composite one can write

$$\frac{1}{\hat{y}_\mu(\hat{\mu}_*)} \leqslant \frac{2\zeta_2}{\kappa_1} + \frac{2\zeta_1}{\kappa_2} + \frac{\eta_2}{\mu_1} + \frac{\eta_1}{\mu_2}. \tag{2.19}$$

The sum of (2.18) and (2.19) gives

$$\frac{1}{y_\mu(\mu_*)} + \frac{1}{\hat{y}_\mu(\hat{\mu}_*)} \leqslant \frac{2}{\kappa_1} + \frac{2}{\kappa_2} + \frac{1}{\mu_1} + \frac{1}{\mu_2}, \tag{2.20}$$

which is equivalent to (2.17). It is interesting to compare the bound (2.20) with the Hashin–Shtrikman–Walpole bound [29, 30] on the shear modulus that can be written in the form

$$\frac{1}{y_\mu(\mu_*)} \leqslant \frac{2}{\kappa_-} + \frac{1}{\mu_-}. \tag{2.21}$$

Each of the terms in (2.21) corresponds to two similar terms in (2.20). As we will see, a similar comparison can be made for the shear modulus bounds for a three-dimensional composite that we will obtain in the next section.

Similarly, one can obtain the upper shear modulus phase-interchange bound by using the geometrical-parameter bounds of Gibiansky and Torquato [13]. However, this relation is weaker than the translation bound (2.15).

Proof of the bounds (2.15) *and* (2.16). To prove the other lower bound (2.16) we need to use translation method and to consider the functional that is the sum of four terms. The first two of them are the values of the energy stored by the composite in two linear independent strain fields with given averages. The other two are corresponding values for the phase-interchanged composite. The lower bound on this functional allows us to obtain bounds on the effective moduli similar to how it was done for the bulk modulus bound. The reader is referred to Refs [18, 20] for further details. Tedious and lengthy algebraic manipulations that we choose to omit (and would not be able to perform without the use of the Maple 5 program [31]) lead to the bound (2.16).

Similarly, to prove the upper bound on the shear modulus we need to consider the functional that is the sum of four terms. The first two of these are the values of the complementary energy stored by the composite in the two linear independent stress fields with a given average. The other two are corresponding values for the phase-interchanged composite. The translation bound on this sum leads to the inequality (2.15).

In the μ_*–$\hat{\mu}_*$ plane, the inequalities (2.15) and (2.16) define a large lens-shaped region. The pairs $(\mu_*, \hat{\mu}_*)$ of all composites lie within this region, *independently of the phase volume fraction.* The bound (2.17) depends on the volume fraction and cuts the edges of this lens-shape region. The inequality (2.16) is always more restrictive than (2.17) in the neighborhood of the point $\mu_*/\hat{\mu}_* = 1$, whereas the expression (2.17) is stronger for large and small ratios of $\mu_*/\hat{\mu}_*$.

It is helpful to specify the bounds for two different cases. First we assume that the phases are well-ordered in the sense that

$$(\kappa_1 - \kappa_2)(\mu_1 - \mu_2) \geqslant 0. \tag{2.22}$$

Then the bounds (2.15) and (2.16) can be rewritten in the form

$$\frac{\mu_1\mu_2 - \mu_*\hat{\mu}_*}{\mu_1 + \mu_2 - \mu_* - \hat{\mu}_*} \leqslant \frac{\mu_1\mu_2(\mu_2\kappa_1 + \mu_1\kappa_2 + 2\mu_2\mu_1)}{\mu_2\kappa_1(\mu_2 + \kappa_2) + 2\mu_1\mu_2(\mu_1 + \mu_2 + \kappa_1 + \kappa_2) + \mu_1\kappa_2(\mu_1 + \kappa_1)}, \tag{2.23}$$

$$\frac{\mu_1\mu_2 - \mu_*\hat{\mu}_*}{\mu_1 + \mu_2 - \mu_* - \hat{\mu}_*} \geqslant \frac{\mu_1\mu_2(\mu_1\kappa_1 + \mu_2\kappa_2 + 2\mu_2\mu_1)}{\mu_2\kappa_2(\mu_2 + \kappa_1) + 2\mu_1\mu_2(\mu_1 + \mu_2 + \kappa_1 + \kappa_2) + \mu_1\kappa_2(\mu_1 + \kappa_2)}, \tag{2.24}$$

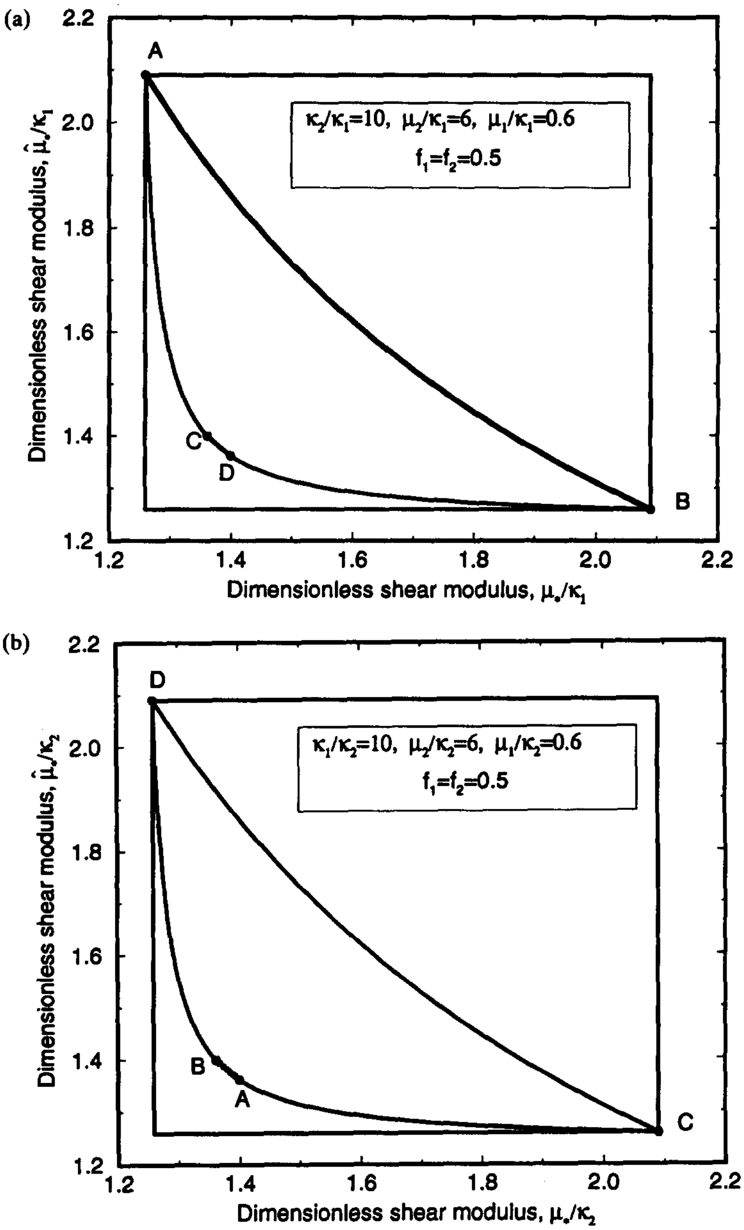

Fig. 2. Shear moduli bounds in the μ_*–$\hat{\mu}_*$ plane for a two-dimensional composite of two well-ordered phases (a) and two badly-ordered phases (b). The rectangles corresponds to the Hashin–Shtrikman–Walpole bounds. The points A and B correspond to the matrix composites. The points C and D correspond to the Walpole bounds on the shear modulus. The bold curves (AB) correspond to the doubly-coated matrix composites.

respectively. Here we have assumed without loss of generality that $\mu_1 \leqslant \mu_2$ and $\kappa_1 \leqslant \kappa_2$. The inequality (2.17) is not sensitive to the sign of the expression (2.22).

These bounds are illustrated in Fig. 2(a) for the choice of the parameters given by (2.10). The rectangle in this figure corresponds to the Hashin–Shtrikman [29] bounds on the shear modulus

$$\mu_{1*} \leqslant \mu_* \leqslant \mu_{2*}, \qquad \hat{\mu}_{2*} \leqslant \hat{\mu}_* \leqslant \hat{\mu}_{1*}, \tag{2.25}$$

where

$$\mu_{1*} = F_\mu(2\kappa_1\mu_1/(\kappa_1+2\mu_1)), \qquad \mu_{2*} = F_\mu(2\kappa_2\mu_2/(\kappa_2+2\mu_2)), \tag{2.26}$$

and

$$\hat{\mu}_{2*} = \hat{F}_\mu(2\kappa_1\mu_1/(\kappa_1+2\mu_1)), \qquad \hat{\mu}_{1*} = \hat{F}_\mu(2\kappa_2\mu_2/(\kappa_2+2\mu_2)). \tag{2.27}$$

There are four characteristic points on the boundary. The lower bound (2.17) and upper bound (2.23) intersect at the points $A = (\mu_{1*}, \hat{\mu}_{1*})$ and $B = (\mu_{2*}, \hat{\mu}_{2*})$. These points correspond to the matrix laminate composites [32] that realize the Hashin–Shtrikman shear modulus bounds. Moreover, in this case the entire upper bound [the bold curve AB in Fig. 2(a)] is also attainable by the type of "doubly-coated matrix composite" similar to ones used by Cherkaev and Gibiansky [5]. Note that these structures also achieve the bulk modulus bounds (2.2). The two lower bounds (2.17) and (2.24) intersect at the point $C(\mu_{3*}, \hat{\mu}_{3*})$.

$$\mu_{3*} = F_\mu(2\kappa_1\mu_2/(\kappa_1 + 2\mu_2)), \qquad \hat{\mu}_{3*} = \hat{F}_\mu(2\kappa_2\mu_1/(\kappa_2 + 2\mu_1)), \tag{2.28}$$

and the point $D = (\mu_{4*}, \hat{\mu}_{4*})$,

$$\mu_{4*} = F_\mu(2\kappa_2\mu_1/(\kappa_2 + 2\mu_1)), \qquad \hat{\mu}_{4*} = \hat{F}_\mu(2\kappa_1\mu_2/(\kappa_1 + 2\mu_2)). \tag{2.29}$$

The translation lower bound (2.24) is stronger than the geometrical parameter bound (2.17) in the interval (CD). The geometrical parameter bound (2.17) is stronger in the intervals (AC) and (DB). The shear modulus bounds also have two corners [see Fig. 2(a)]. The shear modulus $\hat{\mu}_*$ of the phase-interchanged composite with *well-ordered phases* is uniquely defined if the composite shear modulus μ_* is equal to μ_{1*} or μ_{2*}.

For a composite with badly-ordered phases,

$$(\kappa_1 - \kappa_2)(\mu_1 - \mu_2) \leqslant 0, \tag{2.30}$$

the bounds (2.15) and (2.16) can be rewritten in a form similar to (2.23) and (2.24) by substituting $\mu_- = \mu_1$, $\kappa_- = \kappa_2$, $\mu_+ = \mu_2$, $\kappa_+ = \kappa_1$ into these equations if we assume, without loss of generality, that $\mu_1 \leqslant \mu_2$ and $\kappa_1 \geqslant \kappa_2$. As can easily be seen in this case, the bounds have exactly the same form as in the well-ordered case, but the lower bound becomes the upper bound and *vice versa.*

These bounds are illustrated in Fig. 2(b) for the following choice of the parameters:

$$\kappa_1 = 10, \qquad \mu_1 = 0.6, \qquad \kappa_2 = 1, \qquad \mu_2 = 6, \qquad f_1 = f_2 = 0.5, \tag{2.31}$$

which differ from (2.10) by interchanging the values κ_1 and κ_2. In this case, the upper bound (2.15) passes through the points $C = (\mu_{3*}, \hat{\mu}_{3*})$ and $D = (\mu_{4*}, \hat{\mu}_{4*})$, whereas the points $A = (\mu_{1*}, \hat{\mu}_{1*})$ and $B = (\mu_{2*}, \hat{\mu}_{2*})$ lie on the intersection of the lower bounds (2.16) and (2.17) [see Fig. 2(b)]. One can check that the segment (AB) of the lower boundary [the bold line in Fig. 2(b)] is attainable by the doubly-coated matrix laminate composites. The rectangle in Fig. 2(b) corresponds to the Walpole [30] bounds on the shear modulus

$$\mu_{4*} \leqslant \mu_* \leqslant \mu_{3*}, \qquad \hat{\mu}_{3*} \leqslant \hat{\mu}_* \leqslant \hat{\mu}_{4*}. \tag{2.32}$$

The shear modulus $\hat{\mu}_*$ of the composite with *badly-ordered phases* is uniquely defined if the shear modulus μ_* is equal to μ_{3*} or μ_{4*} [see Fig. 2(b)]. Although there is no proof that such composites with $\mu_* = \mu_{3*}$ or $\mu_* = \mu_{4*}$ cannot exist, they still have not been found.

3. PHASE-INTERCHANGE RELATIONS FOR THREE-DIMENSIONAL COMPOSITES

All of the results given here apply to macroscopically isotropic, three-dimensional composites consisting of two isotropic phases.

3.1 Results for the bulk modulus

The following statement specifies the phase-interchange relations for the bulk modulus in three dimensions:

Statement 3. The bulk moduli κ_* and $\hat{\kappa}_*$ of a three-dimensional composite are restricted by the inequalities

$$\frac{f_1 f_2 \kappa_*}{\kappa_*(f_1\kappa_2 + f_2\kappa_1) - \kappa_1\kappa_2} + \frac{f_1 f_2 \hat{\kappa}_*}{\hat{\kappa}_*(f_1\kappa_1 + f_2\kappa_2) - \kappa_1\kappa_2} \leqslant \frac{3\kappa_1\kappa_2(\mu_1+\mu_2)/4 + \mu_1\mu_2(\kappa_1+\kappa_2)}{\mu_1\mu_2(\kappa_1-\kappa_2)^2}, \tag{3.1}$$

$$\frac{f_1 f_2}{f_1\kappa_1 + f_2\kappa_2 - \kappa_*} + \frac{f_1 f_2}{f_1\kappa_2 + f_2\kappa_1 - \hat{\kappa}_*} \leqslant \frac{\kappa_1 + \kappa_2 + 4(\mu_1+\mu_2)/3}{(\kappa_1-\kappa_2)^2}. \tag{3.2}$$

In the κ_*–$\hat{\kappa}_*$ plane, (3.1) is a lower bound and (3.2) is an upper bound.

Proof. In three-dimensions, the geometrical-parameter bounds on the bulk modulus [23] can be presented in the form

$$\left[\frac{3\zeta_1}{4\mu_1} + \frac{3\zeta_2}{4\mu_2}\right]^{-1} \leqslant y_\kappa(\kappa_*) \leqslant \frac{4\zeta_1\mu_1}{3} + \frac{4\zeta_2\mu_2}{3}, \tag{3.3}$$

$$\left[\frac{3\zeta_1}{4\mu_2} + \frac{3\zeta_2}{4\mu_1}\right]^{-1} \leqslant \hat{y}_\kappa(\hat{\kappa}_*) \leqslant \frac{4\zeta_1\mu_2}{3} + \frac{4\zeta_2\mu_1}{3}. \tag{3.4}$$

By taking the sum of the upper bounds in (3.3) and (3.4), we arrive at the inequality

$$y_\kappa(\kappa_*) + \hat{y}_\kappa(\hat{\kappa}_*) \leqslant 4(\mu_1+\mu_2)/3. \tag{3.5}$$

One can check that (3.5) is equivalent to (3.2).

Similarly, taking the sum of the lower bounds (3.3) and (3.4) in the form

$$\frac{1}{y_\kappa(\kappa_*)} \leqslant \frac{3\zeta_1}{4\mu_1} + \frac{3\zeta_2}{4\mu_2}, \qquad \frac{1}{\hat{y}_\kappa(\hat{\kappa}_*)} \leqslant \frac{3\zeta_1}{4\mu_2} + \frac{3\zeta_2}{4\mu_2}, \tag{3.6}$$

we obtain

$$\frac{1}{y_\kappa(\kappa_*)} + \frac{1}{\hat{y}_\kappa(\hat{\kappa}_*)} \leqslant \frac{3}{4\mu_2} + \frac{3}{4\mu_1}, \tag{3.7}$$

which is equivalent to (3.1).

The bounds (3.1) and (3.2) enclose a lens-shaped region in the κ_*–$\hat{\kappa}_*$ plane. The plot of these bounds is very similar to Fig. 1 for the two-dimensional problems, and hence we omit it. As in the two-dimensional case, the bounds (3.1) and (3.2) intersect at the points $(\kappa_{1*}, \hat{\kappa}_{1*})$, and $(\kappa_{2*}, \hat{\kappa}_{2*})$,

$$\kappa_{1*} = F_\kappa(4\mu_1/3), \qquad \hat{\kappa}_{1*} = \hat{F}_\kappa(4\mu_2/3), \tag{3.8}$$

$$\kappa_{2*} = F_\kappa(4\mu_2/3), \qquad \hat{\kappa}_{2*} = \hat{F}_\kappa(4\mu_1/3), \tag{3.9}$$

that correspond to the assemblages of singly-coated spheres that fill the whole space [29]. The point $(\kappa_{1*}, \hat{\kappa}_{1*})$ corresponds to the assemblage where the second phase makes up the included spherical core and the first phase forms the external coating. The point $(\kappa_{2*}, \hat{\kappa}_{2*})$ corresponds to the assemblage where the first phase makes up the core and the second phase forms the external coating.

Unlike the two-dimensional case, neither lower nor upper bounds correspond to the assemblages of space-filling doubly-coated spheres. There exists one point $(\kappa_*^{(0)}, \hat{\kappa}_*^{(0)})$,

$$\kappa_*^{(0)} = F_\kappa\left(\left[\frac{3f_1}{4\mu_2} + \frac{3f_2}{4\mu_1}\right]^{-1}\right), \qquad \hat{\kappa}_*^{(0)} = \hat{F}_\kappa\left(\left[\frac{3f_1}{4\mu_1} + \frac{3f_2}{4\mu_2}\right]^{-1}\right), \tag{3.10}$$

on the lower bound (3.1) that corresponds to the special polycrystal arrangement (see Refs [26, 27]) made of laminates of the two phases.

There exist three points $(\kappa_*^{(i)}, \hat{\kappa}_*^{(i)})$,

$$\kappa_*^{(i)} = F_\kappa(a_i), \qquad \hat{\kappa}_*^{(i)} = \hat{F}_\kappa(\hat{a}_i), \qquad i = 1, 2, 3, \tag{3.11}$$

on the upper bound (3.2) that correspond to specially arranged polycrystals made of the laminates of the two phases ($i = 1$), made of the Hashin [25] assemblages of the coated cylinders where the second phase makes up the core and the first phase makes up the coating ($i = 2$), and made of the Hashin assemblages of complementary coated cylinders ($i = 3$). Here

$$a_1 = \frac{4}{3} f_1 \mu_2 + \frac{4}{3} f_2 \mu_1, \qquad \hat{a}_1 = \frac{4}{3} f_1 \mu_1 + \frac{4}{3} f_2 \mu_2; \tag{3.12}$$

$$a_2 = \frac{4}{3}\left(1 - \frac{f_1}{4}\right)\mu_1 + \frac{4}{3}\frac{f_1}{4}\mu_2, \qquad \hat{a}_2 = \frac{4}{3}\left(1 - \frac{f_1}{4}\right)\mu_2 + \frac{4}{3}\frac{f_1}{4}\mu_1; \tag{3.13}$$

$$a_3 = \frac{4}{3}\frac{f_2}{4}\mu_1 + \frac{4}{3}\left(1 - \frac{f_2}{4}\right)\mu_2, \qquad \hat{a}_3 = \frac{4}{3}\frac{f_2}{4}\mu_2 + \frac{4}{3}\left(1 - \frac{f_2}{4}\right)\mu_1. \tag{3.14}$$

Polycrystal microstructures with the bulk moduli (3.11)–(3.14) are constructed by the procedure suggested by Schulgasser [33] for the conductivity problem. Elastic polycrystals that correspond to these points, and to the point (3.10) have been found by Avellaneda and Milton [26], and Rudelson [27]. Expressions for the bulk moduli of such structures have been obtained by Gibiansky and Milton [19].

As in the two-dimensional case, the region enclosed by the bounds has two narrow corners. It allows us to predict precisely the effective bulk modulus $\hat{\kappa}_*$ of the phase-interchanged composite if the bulk modulus κ_* lies close to the Hashin–Shtrikman bounds κ_{1*} or κ_{2*} and thus, to the Hashin–Shtrikman coated sphere assemblages.

3.2 Results for the shear modulus

Here we give phase-interchange relations for the shear modulus of a three-dimensional composite. First, we need to introduce two functions $L(\eta, \zeta)$ and $U(\eta, \zeta)$

$$L(\eta, \zeta) = F_\mu(\Xi(\eta, \zeta)), \qquad \Xi(\eta, \zeta) = \frac{15\langle \mu^{-1} \rangle_\eta + 48\langle \mu^{-1} \rangle_\zeta + 56\langle \kappa^{-1} \rangle_\zeta}{2\langle \mu^{-1} \rangle_\eta (21\langle \mu^{-1} \rangle_\zeta + 2\langle \kappa^{-1} \rangle_\zeta) + 80\langle \mu^{-1} \rangle_\zeta \langle \kappa^{-1} \rangle_\zeta},$$

$$U(\eta, \zeta) = F_\mu(\Theta(\eta, \zeta)), \qquad \Theta(\eta, \zeta) = \frac{8\langle \mu \rangle_\eta (7\langle \mu \rangle_\zeta + 6\langle \kappa \rangle_\zeta) + 15\langle \mu \rangle_\zeta \langle \kappa \rangle_\zeta}{80\langle \mu \rangle_\eta + 4\langle \mu \rangle_\zeta + 42\langle \kappa \rangle_\zeta}, \tag{3.15}$$

where

$$\langle a \rangle_\eta = \eta a_1 + (1 - \eta) a_2, \qquad \langle a \rangle_\zeta = \zeta a_1 + (1 - \zeta) a_2. \tag{3.16}$$

We denote by $\hat{L}(\eta, \zeta)$ and $\hat{U}(\eta, \zeta)$ the corresponding functions for the phase-interchanged composite. They differ from the functions $L(\eta, \zeta)$ and $U(\eta, \zeta)$ by the obvious change of indices 1 and 2 for the phase moduli.

Statement 4. To find bounds on the shear moduli μ_* and $\hat{\mu}_*$ of a three-dimensional composite we need to consider eight curves in the μ_*–$\hat{\mu}_*$ plane:

$$\begin{aligned}(\mu_*, \hat{\mu}_*) &= (L(1, \zeta), \hat{L}(1, \zeta)), \quad && \zeta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (L(0, \zeta), \hat{L}(0, \zeta)), && \zeta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (L(\eta, 1), \hat{L}(\eta, 1)), && \eta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (L(\eta, 0), \hat{L}(\eta, 0)), && \eta \in [0, 1],\end{aligned} \tag{3.17}$$

$$\begin{aligned}(\mu_*, \hat{\mu}_*) &= (U(1, \zeta), \hat{U}(1, \zeta)), \quad && \zeta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (U(0, \zeta), \hat{U}(0, \zeta)), && \zeta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (U(\eta, 1), \hat{U}(\eta, 1)), && \eta \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (U(\eta, 0), \hat{U}(\eta, 0)), && \eta \in [0, 1],\end{aligned} \tag{3.18}$$

and two additional curves

$$\begin{aligned}(\mu_*, \hat{\mu}_*) &= (L(\eta^L(\zeta), \zeta), \hat{L}(\eta^L(\zeta), \zeta)), \quad && \zeta \in [0, 1], \text{ if } \eta^L(\zeta) \in [0, 1],\\ (\mu_*, \hat{\mu}_*) &= (U(\eta^U(\zeta), \zeta), \hat{U}(\eta^U(\zeta), \zeta)), && \zeta \in [0, 1], \text{ if } \eta^U(\zeta) \in [0, 1],\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\eta^L(\zeta) &= \zeta - \frac{7(\kappa_1\mu_2 - \kappa_2\mu_1)(2\zeta - 1)}{(\mu_1 - \mu_2)(\kappa_1 + \kappa_2 + 3\kappa_1\kappa_2(\mu_1^{-1} + \mu_2^{-1}))},\\ \eta^U(\zeta) &= \zeta + \frac{21(2\zeta - 1)(\kappa_1\mu_2 - \kappa_2\mu_1)}{8(\mu_1 - \mu_2)(\mu_1 + \mu_2 + 3\kappa_1 + 3\kappa_2)}.\end{aligned} \tag{3.20}$$

The envelope or hull formed by these curves give the bounds.

PROOF. First we note that the functions $L(\eta, \zeta)$ and $U(\eta, \zeta)$ represent the upper and lower geometrical-parameter shear modulus bounds of Milton and Phan-Thien [34], namely,

$$L(\eta_1, \zeta_1) \leq \mu_* \leq U(\eta_1, \zeta_1), \qquad \hat{L}(\eta_1, \zeta_1) \leq \hat{\mu}_* \leq \hat{U}(\eta_1, \zeta_1), \tag{3.21}$$

where $\eta_1 = \eta \in [0, 1]$, $\eta_2 = 1 - \eta$ and $\zeta_1 = \zeta \in [0, 1]$, $\zeta_2 = 1 - \zeta$ are the so-called geometrical parameters that involve three-point information about microstructure of the composite. We refer the reader to our recent paper [13] where we have reviewed such known results and have obtained the new representation (3.21) of the Milton–Phan-Thien bounds. We note that the definition of the geometrical parameter η_1 that is used here differs from the original definition used by Milton and Phan-Thien [34].

In terms of Y-transformations, (3.21) can be presented in the form

$$\Xi(\eta_1, \zeta_1) \leq y_\mu(\mu_*) \leq \Theta(\eta_1, \zeta_1), \tag{3.22}$$

where $\Xi(\eta_1, \zeta_1)$ and $\theta(\eta_1, \zeta_1)$ are given in (3.15) (see Ref. [13]). Similarly,

$$\Xi(\eta_2, \zeta_2) \leq \hat{y}_\mu(\mu_*) \leq \Theta(\eta_2, \zeta_2). \tag{3.23}$$

In the $y_\mu(\mu_*)$–$\hat{y}_\mu(\hat{\mu}_*)$ plane the inequalities (3.22) and (3.23) restrict the region where the pair $(y_\mu(\mu_*), \hat{y}_\mu(\hat{\mu}_*))$ must lie for given values of η and ζ. The union of all such regions for $\eta \in [0, 1]$ and $\zeta \in [0, 1]$ obviously contains all the pairs $(y_\mu(\mu_*), \hat{y}_\mu(\hat{\mu}_*))$. Therefore, we need to find the union over all admissible η and ζ of the bounds (3.22), (3.23). To find an upper bound, for any fixed value of $y_\mu(\mu_*) = \Theta(\eta, \zeta)$ we need to find maximal value of $\hat{y}_\mu(\hat{\mu}_*) = \Theta(1 - \eta, 1 - \zeta)$, i.e. solve the following optimization problem:

$$\max_{\substack{\eta \in [0,1], \zeta \in [0,1],\\ \Theta(\eta, \zeta) = y_\mu(\mu_*),}} \Theta(1 - \eta, 1 - \zeta), \tag{3.24}$$

where the maximum is taken over all admissible values of the parameters η and ζ.

To find the lower bound for any fixed value of $y_\mu(\mu_*) = \Xi(\eta, \zeta)$, we need to find minimal value of $\hat{y}_\mu(\hat{\mu}_*) = \Xi(1-\eta, 1-\zeta)$, i.e. solve the minimization problem

$$\min_{\substack{\eta \in [0,1], \zeta \in [0,1], \\ \Xi(\eta, \zeta) = y_\mu(\mu_*).}} \Xi(1-\eta, 1-\zeta). \tag{3.25}$$

Tedious, but straightforward solution of the maximization problem (3.24) and minimization problem (3.25) (that was performed by Maple [31]) leads to the results described by Statement

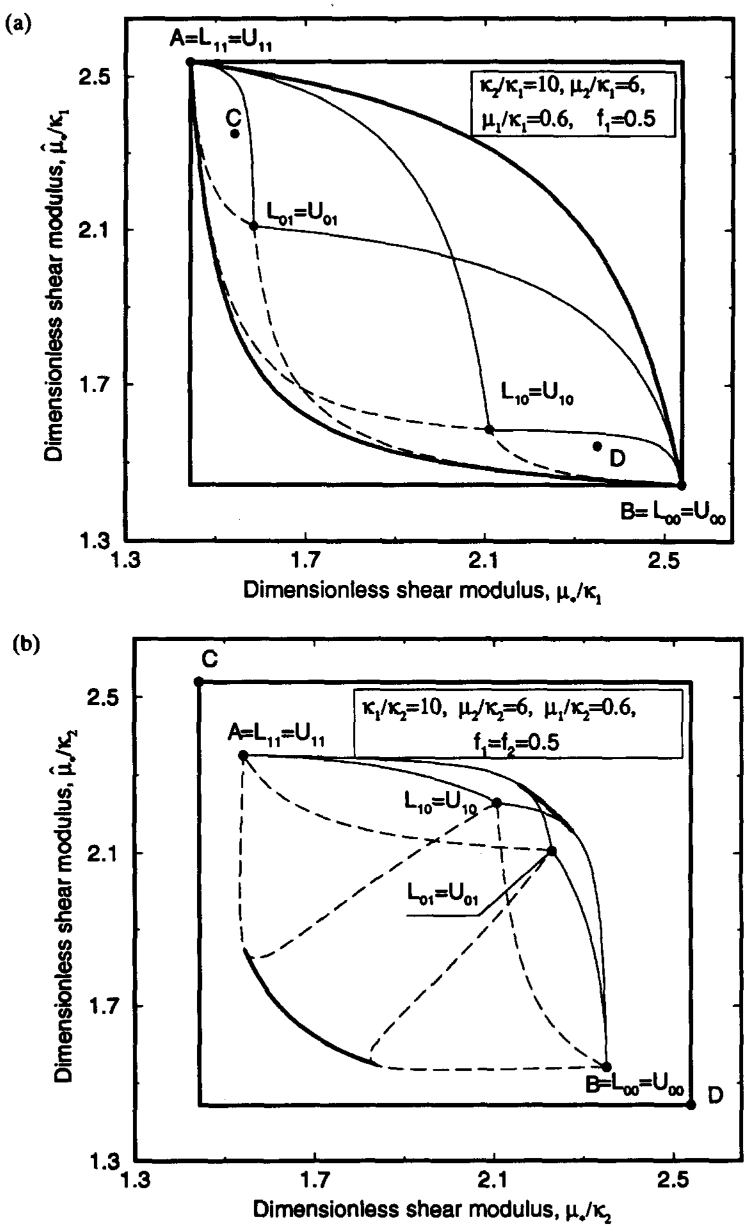

Fig. 3. Shear moduli bounds in the μ_*–$\hat{\mu}_*$ plane for a three-dimensional composite of two well-ordered phases (a) and two badly-ordered phases (b). The rectangles correspond to the Hashin–Shtrikman–Walpole bounds. Two bold curves correspond to the bounds (3.19), the dashed curves are described by the formulas (3.17), the solid curves are defined by the expressions (3.18). The points A and B correspond to the Hashin–Shtrikman bounds on the shear modulus. The points C and D correspond to the Walpole expressions (3.28). The points L_{ij}, U_{ij}, i, $j = 0, 1$ are described in the text. In (a), the bounds (3.19) give almost all of the boundary curves, except for small regions near the points A and B. In (b), the bounds (3.19) give only a fraction of the boundary curves, the rest of the boundary is described by the expressions (3.17) and (3.18).

4. The curves (3.19) correspond to the stationary solutions of the extremal problems (3.24) and (3.25), the curves (3.17) and (3.18) correspond to the values of the parameters η and ζ that lie on the boundary of the admissible intervals $\eta \in [0, 1]$ and $\zeta \in [0, 1]$.

The bounds are illustrated in Fig. 3(a) for a composite with well-ordered phases having the moduli given by (2.10). The rectangle in this figure corresponds to the Hashin–Shtrikman bounds [29] on the shear modulus

$$\mu_{1*} \leq \mu_* \leq \mu_{2*}, \qquad \hat{\mu}_{2*} \leq \hat{\mu}_* \leq \hat{\mu}_{1*}, \tag{3.26}$$

where

$$\mu_{1*} = F_\mu\left(\frac{\mu_1(9\kappa_1 + 8\mu_1)}{6\kappa_1 + 12\mu_1}\right), \qquad \mu_{2*} = F_\mu\left(\frac{\mu_2(9\kappa_2 + 8\mu_2)}{6\kappa_2 + 12\mu_2}\right),$$
$$\hat{\mu}_{2*} = \hat{F}_\mu\left(\frac{\mu_1(9\kappa_1 + 8\mu_1)}{6\kappa_1 + 12\mu_1}\right), \qquad \hat{\mu}_{1*} = \hat{F}_\mu\left(\frac{\mu_2(9\kappa_2 + 8\mu_2)}{6\kappa_2 + 12\mu_2}\right). \tag{3.27}$$

The points $A = (\mu_{1*}, \hat{\mu}_{1*})$ and $B = (\mu_{2*}, \hat{\mu}_{2*})$ correspond to the matrix microstructures that have been found by Francfort and Murat [32]. The points $C = (\mu_{3*}, \hat{\mu}_{3*})$ and $D = (\mu_{4*}, \hat{\mu}_{4*})$ correspond to the Walpole [30] expressions

$$\mu_{3*} = F_\mu\left(\frac{\mu_1(9\kappa_2 + 8\mu_1)}{6\kappa_2 + 12\mu_1}\right), \qquad \hat{\mu}_{3*} = \hat{F}_\mu\left(\frac{\mu_2(9\kappa_1 + 8\mu_2)}{6\kappa_1 + 12\mu_2}\right),$$
$$\mu_{4*} = F_\mu\left(\frac{\mu_2(9\kappa_1 + 8\mu_2)}{6\kappa_1 + 12\mu_2}\right) \qquad \hat{\mu}_{4*} = \hat{F}_\mu\left(\frac{\mu_1(9\kappa_2 + 8\mu_1)}{6\kappa_2 + 12\mu_1}\right). \tag{3.28}$$

The bold solid curves in Fig. 3(a) correspond to the bound (3.19). For this particular choice of the phase moduli, these curves form almost the entire boundary, except for the small regions near the points A and B. The dashed lines in Fig. 3(a) correspond to the curves (3.17), and the thin solid lines correspond to the curves (3.18). We also marked the points L_{00}, L_{01}, L_{10}, L_{11} and U_{00}, U_{01}, U_{10}, U_{11} in Fig. 3(a), where $L_{\eta\zeta} = (L(\eta, \zeta), \hat{L}(\eta, \zeta))$, etc.

The whole set of the points $(\mu_*, \hat{\mu}_*)$ admissible by our bounds is bounded by the hull of the curves (3.17)–(3.19). Again, in this case this set has two sharp corners at the points $A = (\mu_{1*}, \hat{\mu}_{1*})$ and $B = (\mu_{2*}, \hat{\mu}_{2*})$.

Figure 3(b) illustrates our bounds for a composite with badly-ordered phases (2.31). The rectangle in Fig. 3(b) corresponds to the Walpole [30] bounds

$$\mu_{3*} \leq \mu_* \leq \mu_{4*}, \qquad \hat{\mu}_{4*} \leq \hat{\mu}_* \leq \hat{\mu}_{3*}, \tag{3.29}$$

on the moduli μ_* and $\hat{\mu}_*$. One can see that the Walpole bounds can be improved, as was noted earlier by Milton and Phan-Thien [34]. The two bold lines correspond to the bounds (3.19) and (3.20). Unlike the example depicted in Fig. 3(a), the bounds (3.19) form only a fraction of the boundary curves, the rest of the boundary is described by the expressions (3.17) [dashed curves in Fig. 3(b)] and (3.18) [thin solid curves in Figure 3(b)]. Again we marked the characteristic points $L_{00}, \ldots, U_{11}$ in Fig. 3(b).

Note that the bounds described by equations (3.19) have a simple representation in the y_μ–$\hat{y}_\mu$ plane, namely,

$$y_\mu(\mu_*) + \hat{y}_\mu(\hat{\mu}_*) \leq \frac{(\mu_1 + \mu_2)(9\kappa_1 + 9\kappa_2 + 8\mu_1 + 8\mu_2)}{6\kappa_1 + 6\kappa_2 + 12\mu_1 + 12\mu_2}, \tag{3.30}$$

$$\frac{1}{y_\mu(\mu_*)} + \frac{1}{\hat{y}_\mu(\hat{\mu}_*)} \leq \frac{(\mu_1^{-1} + \mu_2^{-1})(12\kappa_1^{-1} + 12\kappa_2^{-1} + 6\mu_1^{-1} + 6\mu_2^{-1})}{8\kappa_1^{-1} + 8\kappa_2^{-1} + 9\mu_1^{-1} + 9\mu_2^{-1}}. \tag{3.31}$$

It is interesting to compare these bounds with the Y-transformation representation of the

Hashin–Shtrikman–Walpole shear modulus bounds that can be written in the form

$$y_\mu(\mu_*) \leq \frac{\mu_+(9\kappa_+ + 8\mu_+)}{6\kappa_+ + 12\mu_+}, \qquad \frac{1}{y_\mu(\mu_*)} \leq \frac{\mu_-^{-1}(12\kappa_-^{-1} + 6\mu_-^{-1})}{8\kappa_-^{-1} + 9\mu_-^{-1}}. \tag{3.32}$$

4. APPLICATIONS TO TWO-DIMENSIONAL COMPOSITES

Our phase-interchange relations are applied here to specific types of two-dimensional composites.

4.1 Limiting cases

Let us now study how our bounds degenerate in different limiting cases:

(i) composites with equal phase bulk moduli (including incompressible limit) or with equal phase shear moduli;

(ii) composites with cavities or a perfectly rigid phase. These results are then applied either to composites near their percolation thresholds or in the low-volume-fraction limit.

First, we note that the lens-shaped region enclosed by our bulk modulus bounds (see Statement 1) collapses into the single point $C = (F_\kappa(\mu), \hat{F}_\kappa(\mu))$ when the phases possess equal shear moduli $\mu_1 = \mu_2 = \mu$. Indeed, for such a choice of the phase moduli, the effective properties of a composite do not depend on the microstructure: the composite is isotropic with the bulk modulus $\kappa_* = F_\kappa(\mu)$ and shear modulus $\mu_* = \mu$ [39].

The shear modulus bounds of Statement 2 allow us to reproduce known phase-interchange relations. Indeed, for a composite of two phases with equal bulk moduli, i.e.

$$\kappa_1 = \kappa_2 = \kappa, \tag{4.1}$$

our upper and lower shear modulus bounds coincide and give the following phase-interchange *equality*:

$$\frac{\mu_1\mu_2 - \mu_*\hat{\mu}_*}{\mu_1 + \mu_2 - \mu_* - \hat{\mu}_*} = \frac{\mu_1\mu_2}{\kappa + \mu_1 + \mu_2}. \tag{4.2}$$

This result is equivalent to the relation (1.8). For incompressible phases, $\kappa = \infty$, it reduces to the equality (1.7).

Consider now two different composites, but with identical microstructures. One of them possesses a phase 1 material with the moduli κ_1 and μ_1 and a *perfectly rigid phase* 2, i.e. $\kappa_2 = \infty$, $\mu_2 = \infty$. We denote the effective moduli of this composite by κ_∞ and μ_∞. The other composite *with the same microstructure* possesses a phase 1 material with the moduli $\hat{\kappa}_1$ and $\hat{\mu}_1$ and a *perfectly soft phase* 2, i.e. $\hat{\kappa}_2 = 0$, $\hat{\mu}_2 = 0$. We denote the effective moduli of this composite κ_0 and μ_0. *It is assumed that the topology of phase* 2 *in these composites is such that it does not percolate.* More exactly, we assume that

$$\kappa_\infty \ll \kappa_2, \qquad \mu_\infty \ll \mu_2, \qquad 1/\kappa_0 \ll 1/\hat{\kappa}_2, \qquad 1/\mu_0 \ll 1/\hat{\mu}_2. \tag{4.3}$$

What are the relationships between κ_∞ and κ_0 and between μ_∞ and μ_0? We will show now that the answer follows directly from the bounds that we obtained in the previous sections.

Consider a composite made of two phases with the moduli κ_1, μ_1 and $\lambda\hat{\kappa}_1$, $\lambda\hat{\mu}_1$, respectively. Here λ is some dimensionless parameter. We will denote the effective moduli of such a composite as κ_λ and μ_λ and the effective moduli of a phase-interchanged composite as $\hat{\kappa}_\lambda$ and $\hat{\mu}_\lambda$. One can easily see that

$$\kappa_\infty = \lim_{\lambda\to\infty} \kappa_\lambda, \qquad \mu_\infty = \lim_{\lambda\to\infty} \mu_\lambda, \qquad \kappa_0 = \lim_{\lambda\to\infty} \frac{\hat{\kappa}_\lambda}{\lambda}, \qquad \mu_0 = \lim_{\lambda\to\infty} \frac{\hat{\mu}_\lambda}{\lambda}. \tag{4.4}$$

Therefore, to get desired relations we need only to evaluate our phase-interchange inequalities in this limit. The following statement summarizes our findings:

Statement 5. For the plane elasticity problem, the effective moduli κ_∞, μ_∞, κ_0 and μ_0 of the aforementioned composites satisfy the upper bounds:

$$\kappa_0\kappa_\infty(\hat\kappa_1+\hat\mu_1)+\kappa_0(\mu_1\hat\mu_1-\kappa_1\hat\kappa_1)-\hat\kappa_1\hat\mu_1(\kappa_1+\mu_1)\leq 0, \tag{4.5}$$

$$\mu_0\mu_\infty(\kappa_1+2\mu_1)(\hat\kappa_1+\hat\mu_1)-\mu_0\mu_1(\kappa_1\hat\mu_1+\hat\kappa_1\mu_1+2\mu_1\hat\mu_1)-\hat\kappa_1\mu_1\hat\mu_1(\kappa_1+\mu_1)\leq 0. \tag{4.6}$$

The lower bounds that can be obtained from the Statements 1 and 2 are trivial in the considered limit in that they degenerate to the appropriate limiting Hashin–Shtrikman lower bounds and hence, are uncoupled.

We shall now apply Statement 5 in two situations: (i) near the percolation threshold and (ii) for small volume fraction of phase 2.

Let us first consider the bounds (4.5) and (4.6) in the vicinity of the percolation threshold of phase 2, i.e. near the point $f_2=f_2^c$ at which the disconnected phase 2 becomes connected. At this point, the moduli κ_∞ and μ_∞ diverge to infinity, and the moduli κ_0 and μ_0 vanish. Then the inequalities (4.5) and (4.6) lead to the following bounds that are valid in the vicinity of the threshold, where the moduli κ_∞, μ_∞, κ_0 and μ_0 still obey the restrictions (4.3)

$$\kappa_0\kappa_\infty(\hat\kappa_1+\hat\mu_1)-\hat\kappa_1\hat\mu_1(\kappa_1+\mu_1)\leq 0, \tag{4.7}$$

$$\mu_0\mu_\infty(\kappa_1+2\mu_1)(\hat\kappa_1+\hat\mu_1)-\hat\kappa_1\mu_1\hat\mu_1(\kappa_1+\mu_1)\leq 0. \tag{4.8}$$

Let us now assume the usual power law behavior of the moduli near the percolation threshold f_2^c, i.e.

$$\kappa_0/\kappa_1=A_0^\kappa(f_2-f_2^c)^{\beta_0},\qquad \mu_0/\mu_1=A_0^\mu(f_2-f_1^c)^{\gamma_0}, \tag{4.9}$$

$$\kappa_\infty/\hat\kappa_1=A_\infty^\kappa(f_2-f_2^c)^{-\beta_\infty},\qquad \mu_\infty/\hat\mu_1=A_\infty^\mu(f_2-f_2^c)^{-\gamma_\infty}. \tag{4.10}$$

We would like to compare the critical exponents β_∞, γ_∞, β_0, γ_0 and amplitudes A_0^κ, A_0^μ, A_∞^κ, A_∞^μ for holes and perfectly rigid inclusions near the threshold. The results follow directly from Statement 5.

Statement 6. For the plane elasticity percolation problems, the critical exponents β_∞, γ_∞, β_0 and γ_0 satisfy the bounds

$$\beta_0\geq\beta_\infty,\qquad \gamma_0\geq\gamma_\infty. \tag{4.11}$$

If

$$\beta_0=\beta_\infty,\qquad \gamma_0=\gamma_\infty, \tag{4.12}$$

then the amplitudes A_0^κ, A_0^μ, A_∞^κ and A_∞^μ satisfy the upper bounds

$$A_0^\kappa A_\infty^\kappa\kappa_1(\hat\kappa_1+\hat\mu_1)-\hat\mu_1(\kappa_1+\mu_1)\leq 0, \tag{4.13}$$

$$A_0^\mu A_\infty^\mu(\kappa_1+2\mu_1)(\hat\kappa_1+\hat\mu_1)-\hat\kappa_1(\kappa_1+\mu_1)\leq 0. \tag{4.14}$$

We now apply Statement 5 in the limit of *low volume fraction of phase* 2. We assume that the volume fraction f_2 is small, and that

$$\kappa_\infty=\kappa_2(1+f_2\alpha_\infty^\kappa),\qquad \mu_\infty=\mu_1(1+f_2\alpha_\infty^\mu),\qquad \kappa_0=\hat\kappa_1(1+f_2\alpha_0^\kappa),\qquad \mu_0=\hat\mu_1(1+f_2\alpha_0^\mu). \tag{4.15}$$

We are interested in the relations among the coefficients α_∞^κ, α_∞^μ, α_0^κ and α_0^μ. One can substitute the relations (4.15) into the bounds (4.5) and (4.6), take the limit $f_2=0$, and arrive at the inequalities

$$\alpha_0^\kappa\hat\mu_1(\kappa_1+\mu_1)+\alpha_\infty^\kappa\kappa_1(\hat\kappa_1+\hat\mu_1)\leq 0, \tag{4.16}$$

$$\alpha_0^\mu\hat\kappa_1(\kappa_1+\mu_1)+\alpha_\infty^\mu(\kappa_1+2\mu_1)(\hat\kappa_1+\hat\mu_1)\leq 0. \tag{4.17}$$

The inequalities (4.5), (4.6) and (4.16), (4.17) become equalities for the matrix laminate composites that correspond to the Hashin–Shtrikman bounds [32]. This means that our bounds are optimal and cannot be improved without additional assumptions about the microstructure. The bounds are tight for composites with effective properties that are close to the Hashin–Shtrikman bounds.

If the phase 1 materials in these composites are the same, as it is in the following examples, we may substitute $\kappa_1 = \hat{\kappa}_1 = \kappa$ and $\mu_1 = \hat{\mu}_1 = \mu$ in all of the formulas of this section.

Application to specific composites. We want to apply the relations (4.13) and (4.14) to specific composites. We will assume here that $\kappa_1 = \hat{\kappa}_1 = \kappa$ and $\mu_1 = \hat{\mu}_1 = \mu$.

As was found by Day *et al.* [35], the elastic moduli of the composite with circular holes on the sites of a *hexagonal* lattice can be described near the threshold $f_2^c = 1 - \pi/3\sqrt{3})$ by the expressions

$$\frac{\kappa_0}{\kappa} = \frac{\sqrt{3}(1-\nu)}{\pi}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{1/2}, \qquad \frac{\mu_0}{\mu} = \frac{\sqrt{3}(1+\nu)}{2\pi}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{1/2}, \tag{4.18}$$

where $\nu = (\kappa - \mu)/(\kappa + \mu)$ is the Poisson's ratio of the matrix. For composites with perfectly rigid circular inclusions with the same matrix phase and a microgeometry as in the case of holes, Chen *et al.* [36] have found that

$$\frac{\kappa_\infty}{\kappa} = \frac{\pi}{\sqrt{3}(1+\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}, \qquad \frac{\mu_\infty}{\mu} = \frac{2\pi}{\sqrt{3}(3-\nu)}\left(\frac{f_2 - f_1^c}{1 - f_2^c}\right)^{-1/2}. \tag{4.19}$$

Taking (4.18) for holes as given information, our bounds (4.13) and (4.14) for the rigid case yield the inequalities

$$\frac{\kappa_\infty}{\kappa} \leqslant \frac{\pi}{\sqrt{3}(1+\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}, \qquad \frac{\mu_\infty}{\mu} \leqslant \frac{2\pi}{\sqrt{3}(3-\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}. \tag{4.20}$$

which remarkably turn out to be exact [cf. (4.19)].

For case of circular holes in the sites of a *triangular* lattice, the behavior near the threshold $f_2^c = 1 - \pi/(2\sqrt{3})$ is described [35] by the expressions

$$\frac{\kappa_0}{\kappa} = \frac{1-\nu}{\pi\sqrt{3}}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{1/2}, \qquad \frac{\mu_0}{\mu} = \frac{4(1+\nu)}{3\sqrt{3}\pi}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{3/2}. \tag{4.21}$$

For the corresponding composite with perfectly rigid circular inclusions [36], the results are

$$\frac{\kappa_\infty}{\kappa} = \frac{\pi\sqrt{3}}{1+\nu}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}, \qquad \frac{\mu_\infty}{\mu} = \frac{\sqrt{3}(3-\nu)\pi}{4(1-\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}. \tag{4.22}$$

Again, if we take (4.21) for holes as given information, our bounds (4.13) and (4.14) for the rigid case give the inequalities

$$\frac{\kappa_\infty}{\kappa} \leqslant \frac{\pi\sqrt{3}}{\pi(1+\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-1/2}, \qquad \frac{\mu_\infty}{\mu} \leqslant \frac{3\pi\sqrt{3}}{4(3-\nu)}\left(\frac{f_2 - f_2^c}{1 - f_2^c}\right)^{-3/2}. \tag{4.23}$$

The upper bulk modulus bound (4.23) is again exact [cf. (4.22)]. The shear modulus bound (4.23) is quite weak, since the critical exponents of the bound (4.23) and the exact result (4.22) are different.

We will now compare our formulas (4.16) and (4.17) to the results obtained by Thorpe *et al.* [37] and Jasiuk [38] who have found the effective elastic moduli of composites with perfectly rigid or perfectly compliant inclusions in the low-volume fraction limit of phase 2.

First we need to reformulate our results in terms of the Young's modulus E and Poisson's ratio ν

$$E = \frac{4\kappa\mu}{\kappa + \mu}, \qquad \nu = \frac{\kappa - \mu}{\kappa + \mu}. \tag{4.24}$$

Table 1. Comparison of the bounds and actual data on the volume-fraction coefficients associated with the effective bulk and shear moduli for polygonal inclusions

Type	α_0^κ	α_0^μ	Bound α_∞^κ	Value α_∞^κ	Bound α_∞^μ	Value α_∞^μ
Triangle	−4.4898	−3.9560	2.4176	1.7664	1.9048	1.9048
Square	−3.3766	−3.4525	1.8182	1.6110	1.6623	1.6266
Pentagon	−3.0952	−3.2705	1.6667	1.5717	1.5747	1.5504
Hexagon	−2.9870	−3.1894	1.6084	1.5566	1.5356	1.5198
Circle	−2.8571	−3.0769	1.5385	1.5385	1.4815	1.4815

In the low-volume fraction limit, the results of Refs [38] and [37] were presented in the form

$$E_0/E = 1 - f\alpha_0^{\mathrm{E}}, \qquad \nu_2 = \nu + f\alpha_0^{\mathrm{E}}(\nu^* - \nu), \tag{4.25}$$

when the phase 2 consists of cavities, and in the form

$$E_\infty/E = 1 + f\alpha_\infty^{\mathrm{E}}, \qquad \nu_\infty = \nu + f\beta, \tag{4.26}$$

when the phase 2 is perfectly rigid. As can be obtained from (4.24),

$$\alpha_0^\kappa = \frac{\alpha_0^E(\nu^* - 1)(\kappa + \mu)}{2\mu}, \qquad \alpha_0^\mu = -\frac{\alpha_0^{\mathrm{E}}(\nu^* + 1)(\kappa + \mu)}{2\kappa}, \tag{4.27}$$

and

$$\alpha_\infty^\kappa = \alpha_\infty^{\mathrm{E}} + \frac{\beta(\kappa + \mu)}{2\mu}, \qquad \alpha_\infty^\mu = \alpha_\infty^{\mathrm{E}} - \frac{\beta(\kappa + \mu)}{2\kappa}. \tag{4.28}$$

Taking the coefficients α_0^{E} and ν^* as given [37], we calculate α_0^κ and α_0^μ via (4.27). Then we apply our bounds (4.16) and (4.17) to get the bounds on α_∞^κ and α_∞^μ. Formulas (4.28) allow us to calculate the exact values for these coefficients by using the data of Refs [37] and [38]. Tables 1 and 2 summarize the comparison of our upper bounds to the exact results for a matrix with the Poisson ratio $\nu = 0.3$ and inclusions of polygonal form (Table 1) or elliptical inclusions with different values of the aspect ratio (Table 2). One can see that our results are in good agreement with the exact results for the polygonal inclusions and elliptical inclusions with aspect ratios near unity. Note that the bounds are tighter for the coefficient associated with shear modulus. The inequality (4.17) becomes an equality for circular inclusions and is virtually identical to the exact equality for triangular inclusions.

For highly "anisotropic" elliptical inclusions with high aspect ratio the bounds are quite wide. This is not surprising because the infinite contrast ratio of the phase properties allows one to achieve a wide range of effective moduli.

4.2 Symmetric composites

Here we apply our bounds to "symmetric" composites, i.e. those in which $\kappa_* = \hat{\kappa}_*$, $\mu_* = \hat{\mu}_*$ and $f_1 = f_2 = 0.5$. Note, however, that we will not use the condition $\mu_* = \hat{\mu}_*$ when we prove the bulk modulus bounds. Similarly, we will not use the condition $\kappa_* = \hat{\kappa}_*$ when proving shear modulus bounds.

Table 2. Comparison of the bounds and actual data on the volume-fraction coefficients associated with the effective bulk and shear moduli for elliptical inclusions

Aspect ratio	α_0^κ	α_0^μ	Bound α_∞^κ	Value α_∞^κ	Bound α_∞^μ	Value α_∞^μ
1.0000	−2.8571	−3.0769	1.5385	1.5385	1.4815	1.4815
3.0000	−4.7619	−4.1026	2.5641	1.8044	1.9753	1.6488
9.0000	−13.0159	−8.5470	7.0085	2.9566	4.1152	2.6239
27.0000	−38.6243	−22.3362	20.7977	6.5316	10.7545	5.9007
81.0000	−115.7319	−63.8557	62.3172	17.2959	30.7453	15.8806

Applications of Statements 1 and 2 to symmetric composites (for which $\kappa_* = \hat{\kappa}_*$, $\mu_* = \hat{\mu}_*$ and $f_1 = f_2$) yields the following results:

Statement 7. The elastic moduli of a *two-dimensional* symmetric composite are bounded by the inequalities

$$\frac{2\kappa_1\kappa_2(\mu_1+\mu_2)+2\mu_1\mu_2(\kappa_1+\kappa_2)}{(\kappa_1+\kappa_2)(\mu_1+\mu_2)+4\mu_1/\mu_2} \leq \kappa_* \leq \frac{\kappa_1\kappa_2-\mu_1\mu_2+\sqrt{(\kappa_1+\mu_1)(\kappa_2+\mu_1)(\kappa_1+\mu_2)(\kappa_2+\mu_2)}}{\kappa_1+\kappa_2+\mu_1+\mu_2}, \tag{4.29}$$

$$A \leq \mu_* \leq B, \quad \text{if} \quad (\kappa_1-\kappa_2)(\mu_1-\mu_2) \geq 0, \tag{4.30}$$

$$B \leq \mu_* \leq A, \quad \text{if} \quad (\kappa_1-\kappa_2)(\mu_1-\mu_2) \leq 0, \tag{4.31}$$

where

$$A = \frac{2\mu_1\mu_2+\mu_1\kappa_1+\mu_2\kappa_2+\sqrt{\mu_1\mu_2(\mu_2+\kappa_1)(\mu_1+\kappa_2)(2+\kappa_1(\mu_1^{-1}+\mu_2^{-1}))(2+\kappa_2(\mu_1^{-1}+\mu_2^{-1}))}}{2(\mu_1+\mu_2+\kappa_1+\kappa_2)+\kappa_1\kappa_2(\mu_1^{-1}+\mu_2^{-1})+\kappa_1\mu_1/\mu_2+\kappa_2\mu_2/\mu_1}, \tag{4.32}$$

$$B = \frac{2\mu_1\mu_2+\mu_1\kappa_2+\mu_2\kappa_1+\sqrt{\mu_1\mu_2(\mu_2+\kappa_1)(\mu_1+\kappa_2)(2+\kappa_1(\mu_1^{-1}+\mu_2^{-1}))(2+\kappa_2(\mu_1^{-1}+\mu_2^{-1}))}}{2(\mu_1+\mu_2+\kappa_1+\kappa_2)+\kappa_1\kappa_2(\mu_1^{-1}+\mu_2^{-1})+\kappa_1\mu_2/\mu_1+\kappa_2\mu_1/\mu_2}. \tag{4.33}$$

The bulk modulus bounds are optimal: there exist polycrystal constructions (described in Section 1) that realize the lower bound (4.29) and there exist assemblages of doubly-coated circles that realize the upper bound (4.29). Note that the upper bound (4.29) is valid independently of volume fraction provided the condition $\kappa_* = \hat{\kappa}_*$ holds. Both upper and lower shear modulus bounds are valid independently of the volume fraction provided the condition $\mu_* = \hat{\mu}_*$ holds. One of these bounds is optimal, namely, there exist doubly-coated matrix composites [5] that deliver equality to the upper bound (4.30). The same composites achieve the lower bound (4.31).

Let us now consider symmetric composites with a high phase-contrast ratio. We assume that the moduli κ_1, μ_1 are finite and moduli κ_2, μ_2 are of the order of δ, where δ is a small parameter. Statement 7 leads to the following asymptotic expressions.

Statement 8. The bulk and shear moduli of a two-dimensional symmetric material with high-contrast phases are bounded by the inequalities

$$2(\kappa_2+\mu_2) \leq \kappa_* \leq \sqrt{\kappa_1\mu_1\frac{\kappa_2+\mu_2}{\kappa_1+\mu_1}}, \tag{4.34}$$

$$\mu_2+\sqrt{\mu_2(\kappa_2+2\mu_1)} \leq \mu_* \leq \sqrt{\frac{\kappa_1\mu_1\mu_2(\kappa_2+\mu_2)}{(\kappa_1+\mu_1)(\kappa_2+2\mu_2)}}. \tag{4.35}$$

One can see that the stiffness of the order of $\sqrt{\delta}$ is the *maximal* one that one can achieve for the bulk and shear moduli of a two-dimensional symmetric composite. This is in agreement with the results by Berlyand and Kozlov [40] who studied the elastic moduli of a checkerboard. They recovered, in particular, that the moduli of such a composite (which is symmetric by construction) are of the order of $\sqrt{\delta}$. The stress fields in the vicinity of the corners of the checkerboard cells (referred to by them as "choke points") are responsible for the square-root dependence of the effective bulk modulus. Although the checkerboard is not isotropic, but only square symmetric, one can make it isotropic by making a polycrystal. Also our bulk modulus results are applicable to the symmetric material with square symmetry of effective stiffness tensor (or cubic symmetry of effective stiffness tensor in three dimensions).

Another example of the symmetric construction that realizes the upper bound (4.34) exactly is an assemblage of the doubly-coated circular inclusions. One should note an important difference between these types of symmetric composites. The checkerboard gives an example of a *geometrically* symmetric composite. Obviously, the effective properties of the two checkerboard-like composites with phase-interchanged positions of the black and white phases are equal independently of the phases moduli. In contrast, a "symmetric" assemblage of the doubly coated spheres is not geometrically symmetric. Structural parameters of such a composite should be chosen precisely in order to satisfy the symmetry condition $\kappa_* = \hat{\kappa}_*$. Interestingly, the condition $\kappa_* = \hat{\kappa}_*$ defines the upper bound uniquely, even for an arbitrary volume fraction of the phases $f_1 = 1 - f_2 \neq 1/2$. The same is true for the shear modulus bounds (4.35). The upper shear modulus bound (4.35) is optimal as it corresponds to the doubly-coated matrix composites.

5. APPLICATION TO THREE-DIMENSIONAL COMPOSITES

Here we apply the phase-interchange relations as we did for two-dimensional composites, except for the case of composites with cavities or a perfectly rigid phase. Bounds in these latter cases are trivial in that they degenerate to the appropriate limiting Hashin–Shtrikman upper and lower bounds and hence, are uncoupled.

5.1 Limiting cases

As in two dimensions, the bulk modulus bounds of Statement 3 collapse onto the single point

$$\kappa_* = F_\kappa(4\mu/3), \qquad \hat{\kappa}_* = \hat{F}_\kappa(4\mu/3) \tag{5.1}$$

in the limit $\mu_1 = \mu_2 = \mu$. This is in agreement with the result by Hill [41] that the composite built from the phases with equal shear moduli is isotropic with the bulk modulus (5.1) and shear modulus $\mu_* = \mu$.

Unlike the two-dimensional problem, the shear modulus phase-interchange bounds in three dimensions do not coincide, even when the phase bulk moduli have the same value. However they can be simplified for a composite consisting of incompressible phases, i.e.

$$\kappa_1 = \kappa_2 = \infty. \tag{5.2}$$

Statement 9. The shear moduli μ_* and $\hat{\mu}_*$ of a three-dimensional composite with incompressible phases are restricted by the inequalities

$$\frac{f_1 f_2 \mu_*}{\mu_*(f_1\mu_2 + f_2\mu_1) - \mu_1\mu_2} + \frac{f_1 f_2 \hat{\mu}_*}{\hat{\mu}_*(f_1\mu_1 + f_2\mu_2) - \mu_1\mu_2} \leqslant \frac{5(\mu_1+\mu_2)}{3(\mu_1-\mu_2)^2}, \tag{5.3}$$

$$\frac{f_1 f_2}{f_1\mu_1 + f_2\mu_2 - \mu_*} + \frac{f_1 f_2}{f_2\mu_1 + f_1\mu_2 - \hat{\mu}_*} \leqslant \frac{5(\mu_1+\mu_2)}{2(\mu_1-\mu_2)^2}. \tag{5.4}$$

In the μ_*–$\hat{\mu}_*$ plane, (5.3) is a lower bound and (5.4) is an upper bound.

Proof. To prove the upper bound (5.4), we note that the bounds (3.22) for such a composite can be rewritten in a form

$$\left(\frac{5\eta_1}{14\mu_1} + \frac{5\eta_2}{14\mu_2}\right)^{-1} + \left(\frac{8\zeta_1}{7\mu_1} + \frac{8\zeta_2}{7\mu_2}\right)^{-1} \leqslant y_\mu(\mu_*) \leqslant \frac{5\eta_1\mu_1}{14} + \frac{5\eta_2\mu_2}{14} + \frac{8\zeta_1\mu_1}{7} + \frac{8\zeta_2\mu_2}{7}. \tag{5.5}$$

Similarly, for the phase-interchanged composite,

$$\left(\frac{5\eta_1}{14\mu_2} + \frac{5\eta_2}{14\mu_1}\right)^{-1} + \left(\frac{8\zeta_1}{7\mu_2} + \frac{8\zeta_2}{7\mu_1}\right)^{-1} \leqslant \hat{y}_\mu(\hat{\mu}_*) \leqslant \frac{5\eta_1\mu_2}{14} + \frac{5\eta_2\mu_1}{14} + \frac{8\zeta_1\mu_2}{7} + \frac{8\zeta_2\mu_1}{7}. \tag{5.6}$$

Taking the sum of the upper bounds in (5.5) and (5.6) we obtain

$$y_\mu(\mu_*) + \hat{y}_\mu(\hat{\mu}_*) \leq \frac{3}{2}(\mu_1 + \mu_2), \tag{5.7}$$

which is equivalent to (5.4).

To prove the lower bound (5.3), we note that for the incompressible phases the equation (3.20) gives

$$\eta^L(\zeta) = \zeta. \tag{5.8}$$

Therefore, the condition $\eta^L \in [0, 1]$ is always satisfied, the first curve in (3.19) gives the entire lower bound. As follows from (3.31) it can be written in the form

$$\frac{1}{y_\mu(\mu_*)} + \frac{1}{\hat{y}_\mu(\hat{\mu}_*)} \leq \frac{2}{3\mu_1} + \frac{2}{3\mu_2}, \tag{5.9}$$

which is equivalent to (5.3).

5.2 Symmetric composites

For the elastic moduli of three-dimensional symmetric composite with $\kappa_* = \hat{\kappa}_*$, $\mu_* = \hat{\mu}_*$, and $f_1 = f_2$. Statements 3 and 4 yield the following bounds:

Statement 10. The bulk and shear moduli of a *three-dimensional* symmetric composite are bounded by the inequalities

$$\frac{6\kappa_1\kappa_2(\mu_1+\mu_2) + 8\mu_1\mu_2(\kappa_1+\kappa_2)}{3(\kappa_1+\kappa_2)(\mu_1+\mu_2) + 16\mu_1\mu_2} \leq \kappa_* \leq \frac{6\kappa_1\kappa_2 + 2(\kappa_1+\kappa_2)(\mu_1+\mu_2)}{3(\kappa_1+\kappa_2) + 4(\mu_1+\mu_2)}, \tag{5.10}$$

$$\frac{5\mu_1\mu_2(\mu_1+\mu_2)(4\kappa_1^{-1} + 4\kappa_2^{-1} + 3\mu_1^{-1} + 3\mu_2^{-1})}{3(\mu_1^{-1}+\mu_2^{-1})(\mu_1^2+\mu_2^2+8\mu_1\mu_2) + 2(\kappa_1^{-1}+\kappa_2^{-1})(3\mu_1^2+3\mu_2^2+14\mu_1\mu_2)}$$
$$\leq \mu_* \leq \frac{\mu_1+\mu_2}{2} - \frac{3(\mu_1-\mu_2)^2(\kappa_1+\kappa_2+2\mu_1+2\mu_2)}{5(\mu_1+\mu_2)(3\kappa_1+3\kappa_2+4\mu_1+4\mu_2)}. \tag{5.11}$$

The bulk modulus bounds are optimal, i.e. there exist polycrystal constructions [described by the equations (3.10)–(3.12)] that realize the upper and lower bounds (5.10).

Immediately from this statement we deduce the bounds for a symmetric composite with incompressible phases.

Statement 11. The shear modulus of a three-dimensional symmetric composite with incompressible phases $\kappa_1 = \kappa_2 = \infty$ is bounded by the inequalities

$$\frac{5\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_2^2+8\mu_1\mu_2} \leq \mu_* \leq \frac{3\mu_1^2+3\mu_2^2+14\mu_1\mu_2}{10\mu_1+10\mu_2}. \tag{5.12}$$

Let us now consider three-dimensional symmetric composites with high phase-contrast ratio. As in the two-dimensional problem we assume that the moduli κ_1, μ_1 are finite and moduli κ_2, μ_2 are of the order of δ, where δ is a small parameter. Statement 10 leads to the following asymptotic expressions.

Statement 12. The bulk and shear moduli of a three-dimensional symmetric material with high-contrast phases are bounded by the inequalities

$$2(\kappa_2 + 4\mu_2/3) \leq \kappa_* \leq \frac{2\kappa_1\mu_1}{3\kappa_1 + 4\mu_1}, \tag{5.13}$$

$$\frac{5\mu_2(3\kappa_1+4\mu_2)}{3\kappa_2+6\mu_2} \leq \mu_* \leq \frac{\mu_1(9\kappa_1+8\mu_1)}{30\kappa_1+40\mu_1}. \tag{5.14}$$

As can be seen, the moduli of a three-dimensional symmetric material can be of the order of the elastic moduli of the stiff phase. For example, a symmetric composite structure that contains rods of both phases aligned in three orthogonal directions possesses a bulk modulus of the order of κ_1 or μ_1 of the stiff phase. A similar system of rods aligned in six directions provides a shear modulus of the order of the elastic moduli of the stiff phase. Clearly, such constructions cannot exist in two-dimensions. It is clear that the fundamental topological difference between two- and three-dimensional spaces lead to the principal difference in the bounds (4.34) and (4.35) and (5.13) and (5.14).

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