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BOUNDS ON THE EFFECTIVE MODULI OF CRACKED MATERIALS

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ABSTRACT

We find bounds on the effective elastic moduli of cracked materials in terms of the effective conductivity of such media. These represent the first non-trivial bounds on the effective properties of cracked media which are independent of the shapes and spatial distribution of the cracks. Different approximations for the elastic moduli of cracked media are tested against our bounds. The microgeometries of cracks that satisfy the bounds exactly are identified.

1. INTRODUCTION

We consider a homogenization problem for a material weakened by cracks. Cracks in the elastic body can be considered to be a particular limiting case of inclusions in an elastic matrix in the limit when the elastic moduli of the inclusions tends to zero and their volume fraction also tends to zero. In this distinguished limit, all conventional bounds on the effective properties of heterogeneous media fail to deliver any useful results. For example, consider the Hashin–Shtrikman (1963) bounds on the effective bulk modulus κ_* of a three-dimensional composite

$$f_1\kappa_1+f_2\kappa_2-\frac{f_1f_2(\kappa_1-\kappa_2)^2}{f_1\kappa_2+f_2\kappa_1+4\mu_2/3}\leqslant\kappa_*\leqslant f_1\kappa_1+f_2\kappa_2-\frac{f_1f_2(\kappa_1-\kappa_2)^2}{f_1\kappa_2+f_2\kappa_1+4\mu_1/3}. \tag{1.1}$$

Here κ_1, κ_2, μ_1, and μ_2 ($\mu_2\leqslant\mu_1$) are the phase bulk and shear moduli, and f_1 and f_2 are the phase volume fractions. Consider the limit when the quantities κ_2, μ_2, and f_2 all tend to zero (i.e. phase 2 forms the cracks). Then the upper bound in (1.1) degenerates into the trivial one

$$\kappa_*\leqslant\kappa_1, \tag{1.2}$$

whereas the lower bound

$$\kappa_* \geqslant \kappa_1 - \frac{\kappa_1^2}{\kappa_1 + (\kappa_2 + 4\mu_2/3)/f_2} \tag{1.3}$$

is indeterminate since it involves the indeterminate ratio $(\kappa_2+4\mu_2/3)/f_2$.

The effective properties of cracked media depend on the so-called crack density ρ that involves the number of cracks per unit volume and their shape characteristics [see paper by Kachanov (1992) for details]. The crack density plays a role roughly similar to the role of the volume fraction for two-phase composites. However, unlike the two-phase composite case, there exist no geometrically independent (i.e. valid for any arrangement of cracks) bounds on the elastic moduli of a cracked material in terms of the crack density. This was pointed out by Kachanov (1992) who presented an example of a low-density crack configuration that can make the effective moduli arbitrarily small, and another example where a high-density crack configuration has a negligible effect on the effective moduli.

In this paper, we obtain the first bounds on the effective moduli of a cracked material that are independent of the shapes and spatial distribution of the cracks. Specifically, we bound the effective elastic moduli of a cracked medium in terms of the effective conductivity of this medium. This is accomplished by applying our recently derived cross-property bounds that link the effective elastic moduli of two-phase composites to the effective conductivity of such a material [see Gibiansky and Torquato (1993, 1995a,b)]. Unlike similar bounds of Berryman and Milton (1988), our new cross-property bounds do not diverge when the contrast between the phases is very large. Moreover, these bounds are valid for arbitrary volume fractions, thus allowing us to avoid difficulties with the zero-volume-fraction limit for the crack problem.

We note that the results are obtained by rigorous examination of the two-phase composite material with one of the phases being ideal. Our analyses provides a bridge between homogenization theory of two-phase composites and crack theory.

In Section 2 we state rigorous upper bounds on the elastic moduli of a cracked material in terms of the effective conductivity of this material that are valid for arbitrary shapes and spatial distribution of the cracks. In Section 3 we compare our bounds with different approximations for the effective moduli of cracked materials. In Section 4 we make concluding remarks.

2. BOUNDS

Consider the cracked material to be a porous-matrix composite. The matrix has conductivity σ, bulk modulus κ, and shear modulus μ. An elastic isotropic material can be also characterized by its Young modulus E and Poisson's ratio ν. The following expressions give the connection between these constants and the bulk and shear moduli

$$\kappa = \frac{E}{2(1-\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad \nu = \frac{\kappa-\mu}{\kappa+\mu}, \quad (d=2), \tag{2.1}$$

$$\kappa = \frac{E}{3(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad \nu = \frac{3\kappa-2\mu}{6\kappa+2\mu}, \quad (d=3), \tag{2.2}$$

where d is the spatial dimension. The pores or cracks can be viewed as inclusions of an ideal phase with conductivity $\sigma_p = 0$, bulk modulus $\kappa_p = 0$, and shear modulus $\mu_p = 0$. We assume that the cracked material is macroscopically isotropic and characterized by the effective bulk modulus κ_*, shear modulus μ_*, and conductivity σ_*. Alternatively, the stiffness of the composite can be characterized by the effective Young's modulus E_* and Poisson's ratio ν_*.

The upper bounds of Gibiansky and Torquato (1993, 1995a,b) for a composite with an arbitrary porosity are applicable to cracked media and read as follows:

Statement 1. The effective moduli of cracked media for $d = 2$ and $d = 3$ satisfy the inequalities

$$\left(\frac{1}{\kappa_*} - \frac{1}{\kappa}\right) \geqslant \frac{(\kappa+\mu)\sigma}{2\kappa\mu}\left(\frac{1}{\sigma_*} - \frac{1}{\sigma}\right), \quad (d=2), \tag{2.3}$$

$$\left(\frac{1}{\mu_*} - \frac{1}{\mu}\right) \geqslant \frac{(\kappa+\mu)\sigma}{\kappa\mu}\left(\frac{1}{\sigma_*} - \frac{1}{\sigma}\right), \quad (d=2), \tag{2.4}$$

$$\left(\frac{1}{E_*} - \frac{1}{E}\right) \geqslant \frac{3\sigma}{2E}\left(\frac{1}{\sigma_*} - \frac{1}{\sigma}\right), \quad (d=2), \tag{2.5}$$

$$\left(\frac{1}{\kappa_*} - \frac{1}{\kappa}\right) \geqslant \frac{3\sigma}{2\mu}\min\left\{1, \frac{1-\nu}{1+\nu}\right\}\left(\frac{1}{\sigma_*} - \frac{1}{\sigma}\right), \quad (d=3), \tag{2.6}$$

which are independent of the shapes and spatial distribution of the cracks.

The corresponding lower bounds on the effective elastic moduli are trivially equal to zero. The inequalities (2.3), (2.4), and (2.6) of this statement are nothing more than particular limits of the conductivity–elastic moduli bounds obtained by Gibiansky and Torquato (1993, 1995a,b). The inequality (2.5) is an immediate corollary of (2.3) and (2.4).

All of the bounds of Statement 1 are optimal, i.e. they correspond to particular structures and cannot be improved on without additional information about crack shapes and distribution. One can easily check that the upper bulk modulus bound (2.3) corresponds to the effective moduli of space-filling assemblages of coated circles (Hashin, 1988) where the core phase is a void or pore phase. One can fill these pores with the same matrix material (but leaving a crack between the outermost coating and the internal circle) without changing the effective moduli of such a medium. The core does not affect the effective properties, because it is separated from the main matrix by the internal circular crack. Thus, the bulk modulus bound (2.3) is valid as an equality for space-filling assemblages of circles with a circular crack in each of them. By changing the ratio of the radii of the cracks and circles one can get composites that correspond to any point on the boundary (2.3). Corresponding microstructures in three dimensions (Hashin and Shtrikman, 1963) achieve the equality sign in the bulk modulus bound (2.6). The only difference is that space-filling assemblages of

circles with circular cracks are replaced by space-filling assemblages of spheres with spherical cracks. The relations (2.3)–(2.5) are valid as equalities for porous matrix laminate composites (Francfort and Murat, 1986) that achieve the Hashin–Shtrikman upper bounds on the effective conductivity, and bulk and shear moduli. As we will see below, the two-dimensional shear modulus bound (2.4) also corresponds to the non-interacting cracks approximation (Bristow, 1960; Kachanov, 1992) for a material with randomly distributed linear cracks.

We note that the bounds (2.3)–(2.6) require some additional assumptions regarding the moduli of the ideal phase. Specifically, we assume that for the two-dimensional problem

$$\frac{(\kappa+\mu)\sigma}{2\kappa\mu} \leqslant \frac{2\sigma_p}{\kappa_p+2\mu_p}, \quad d=2, \tag{2.7}$$

and for the three-dimensional problem

$$\min\left\{1, \frac{1-\nu}{1+\nu}\right\}\frac{\sigma_1}{\mu_1} \leqslant \frac{6\sigma_p}{3\kappa_p+4\mu_p}, \quad d=3. \tag{2.8}$$

The values of the parameters κ_p, μ_p, and σ_p, are equal to zero in the considered limit, but to obtain the bound of Statement 1 we need to assume that the ratios on the right-hand sides of (2.7) and (2.8) lie within the specified limits. The reader is referred to the papers by Gibiansky and Torquato (1995a,b) for details.

At the moment, we do not have appropriate conductivity–shear modulus bounds for three-dimensional composites. All of the known bounds diverge in the limit that we are interested in, and therefore cannot lead to useful shear modulus bounds for cracked bodies.

Remark. There is only one other known conductivity–elastic moduli bound that does not degenerate for the case of cracks, namely, the relation found by Milton (1984):

$$\kappa_*/\kappa \leqslant \sigma_*/\sigma, \quad d=2 \quad \text{or} \quad d=3. \tag{2.9}$$

The inequality (2.9) is also valid in the limit of a non-conducting void phase and provides a meaningful bound on the effective bulk modulus of a cracked body. In general it is weaker than (2.3) or (2.6) and coincides with them only if the Poisson's ratio of the matrix material is equal to zero. The proof of relation (2.9) also requires additional assumptions that the Poisson's ratios of the matrix material and cracks are non-negative, i.e.

$$\nu \geqslant 0, \quad \nu_p \geqslant 0, \tag{2.10}$$

and also that

$$\kappa_p/\sigma_p \leqslant \kappa/\sigma. \tag{2.11}$$

We note that others have attempted to establish a connection between the conductivity and elastic properties of cracked bodies. For example, Bristow (1960) actually related the decrease in the effective elastic moduli due to cracking in an elastic material to the decrease in the effective conductivity of such a material. He did so by

obtaining approximations (non-interacting cracks approximation described below) for the effective conductivity and elastic moduli of cracked materials. But this and a few other similar results in this direction are concerned with materials having cracks with specific shapes and a spatial distribution. By contrast, our bounds are valid for cracks of arbitrary shapes and distribution.

We shall now apply our bounds in cases where the phase moduli of the matrix of a cracked material are unknown. Let us assume that we can measure the effective moduli κ, μ, and σ of this material. Moreover, let us assume that additional microcracks form in this body under some loading, so that its effective moduli change and at some point in time are given by κ_*, μ_*, and σ_*, respectively. We further assume that the differences $d\kappa = \kappa_* - \kappa$, $d\mu = \mu_* - \mu$, and $d\sigma = \sigma_* - \sigma$ are small. We are interested in the relations between the differentials $d\kappa$, $d\mu$, and $d\sigma$.

One possible, although not absolutely rigorous, way to find such relations is to treat the original material with moduli κ, μ, and σ as homogeneous, and consider the differences $d\kappa$, $d\mu$, and $d\sigma$ as resulting from microcracking of this original "homogeneous" material. Essentially, this is similar to the assumption of the differential scheme approximation where one treats the material obtained at the previous stage as a homogeneous isotropic material and mixes it with a small amount of inclusions (or cracks in our case). Under such assumptions, we immediately have from the Statement 1 the following inequalities

$$d\kappa \leqslant \frac{(\kappa+\mu)\kappa}{2\mu\sigma} d\sigma, \quad (d = 2), \tag{2.12}$$

$$d\mu \leqslant \frac{(\kappa+\mu)\mu}{2\kappa\sigma} d\sigma, \quad (d = 2), \tag{2.13}$$

$$dE \leqslant \frac{3E\sigma}{2} d\sigma, \quad (d = 2), \tag{2.14}$$

$$d\kappa \leqslant \frac{3\kappa^2}{2\mu\sigma} \min\left\{1, \frac{1-\nu}{1+\nu}\right\} d\sigma, \quad (d = 3). \tag{2.15}$$

Here E and E_* are the Young's moduli of the original material and the material with additional cracks, and $dE = E_* - E$. We would like to emphasize that unlike the bounds of Statement 1, the bounds in the form of (2.12)–(2.15) are derived under the assumption that we may treat the original cracked material as homogeneous.

3. APPROXIMATIONS

The bounds of Statement 1 are applicable for arbitrary shapes and configuration of the cracks. In this section we test them against different approximation schemes for the effective moduli of material with statistically isotropic distributions of randomly oriented, linear cracks ($d = 2$) or penny-shaped cracks ($d = 3$). We present three different approximations available in the literature. All of these results are discussed in detail in the excellent review article by Kachanov (1992).

3.1. *Self-consistent scheme*

The self-consistent approximation for the effective moduli of a two-dimensional body with linear cracks [see Hill (1965) and Budiansky and O'Connell (1976)] is given by

$$\left(\frac{1}{\kappa_*}-\frac{1}{\kappa}\right)=\frac{\kappa+\mu}{2\kappa\mu}\frac{\pi\rho}{1-\pi\rho},\quad \left(\frac{1}{\mu_*}-\frac{1}{\mu}\right)=\frac{\kappa+\mu}{2\kappa\mu}\frac{\pi\rho}{1-\pi\rho},\tag{3.1}$$

$$\left(\frac{1}{E_*}-\frac{1}{E}\right)=\frac{1}{E}\frac{\pi\rho}{1-\pi\rho},\quad \left(\frac{1}{\sigma_*}-\frac{1}{\sigma}\right)=\frac{1}{\sigma}\frac{\pi\rho}{2-\pi\rho}.\tag{3.2}$$

Here

$$\rho=\frac{1}{A}\Sigma(l^{(i)})^2\tag{3.3}$$

is the crack density, A is the representative area, $2l^i$ is the length of the ith crack, and the sum is taken over all cracks in the region A.

For $d=3$, this scheme yields two algebraic equations for κ_* and μ_* which must be solved numerically.

3.2. *Differential scheme*

The differential scheme for a two-dimensional material with linear cracks (Hashin, 1988) yields

$$\left(\frac{1}{\kappa_*}-\frac{1}{\kappa}\right)=\frac{\kappa+\mu}{2\kappa\mu}(e^{\pi\rho}-1),\quad \left(\frac{1}{\mu_*}-\frac{1}{\mu}\right)=\frac{\kappa+\mu}{2\kappa\mu}(e^{\pi\rho}-1),\tag{3.4}$$

$$\left(\frac{1}{E_*}-\frac{1}{E}\right)=\frac{1}{E}(e^{\pi\rho}-1),\quad \left(\frac{1}{\sigma_*}-\frac{1}{\sigma}\right)=\frac{1}{\sigma}(e^{\pi\rho/2}-1).\tag{3.5}$$

Again for $d=3$, this scheme requires a numerical solution.

3.3. *Non-interacting cracks*

In two dimensions, the non-interacting cracks approximation for the effective moduli of a two-dimensional material with randomly distributed linear cracks (Bristow, 1960) leads to the formulae

$$\left(\frac{1}{\kappa_*}-\frac{1}{\kappa}\right)=\frac{(\kappa+\mu)}{2\kappa\mu}\pi\rho,\quad \left(\frac{1}{\mu_*}-\frac{1}{\mu}\right)=\frac{(\kappa+\mu)}{2\kappa\mu}\pi\rho,\tag{3.6}$$

$$\left(\frac{1}{E_*}-\frac{1}{E}\right)=\frac{1}{E}\pi\rho,\quad \left(\frac{1}{\sigma_*}-\frac{1}{\sigma}\right)=\frac{1}{2\sigma}\pi\rho.\tag{3.7}$$

In three dimensions, this approximation for a material with randomly distributed penny-shaped cracks (Bristow, 1960) gives

$$\left(\frac{1}{\kappa_*}-\frac{1}{\kappa}\right)=\frac{4(3\kappa+4\mu)}{3\mu(3\kappa+\mu)}\rho,\quad \left(\frac{1}{\mu_*}-\frac{1}{\mu}\right)=\frac{16(9\kappa+4\mu)(3\kappa+4\mu)}{45\mu(3\kappa+2\mu)(3\kappa+\mu)}\rho,\quad \left(\frac{1}{\sigma_*}-\frac{1}{\sigma}\right)=\frac{8}{9\sigma}\rho, \tag{3.8}$$

where

$$\rho=\frac{1}{V}\Sigma(l^{(i)})^3 \tag{3.9}$$

is the crack density, V is the representative volume, l^i is the radius of the ith crack, and the sum is taken over all cracks in the volume V.

3.4. *Comparison of bounds and approximations*

It is seen that none of the approximations violate either the bounds of Statement 1 or the bounds in the differential form (2.12)–(2.15). Moreover, the shear modulus bound (2.4) coincides with the non-interacting cracks approximation and hence is an optimal bound. It cannot be improved because it corresponds to a particular microstructure, at least for crack arrangements consistent with the assumptions that make the non-interacting cracks approximation valid.

Figure 1 illustrates the bulk modulus bound (2.3), Milton bound (2.9), and approximations for a two-dimensional cracked material with a matrix Poisson's ratio $\nu = 1/3$.

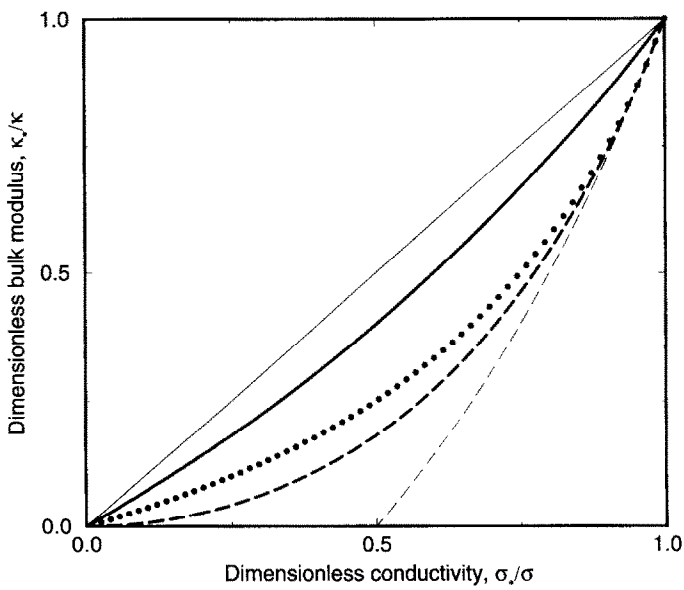

Fig. 1. Comparison of cross-property upper bulk modulus bound (2.3) (bold curve) and Milton bound (2.9) (light straight line), that are valid for arbitrary shape and spatial distribution of cracks, to bulk modulus approximation schemes, for two-dimensional composites with randomly distributed linear cracks. Self-consistent scheme (3.1), (3.2) (light dashed curve), differential scheme (3.4), (3.5) (bold dashed curve), and non-interacting cracks (3.6), (3.7) (dotted curve). The lower bound is trivial and equal to zero.

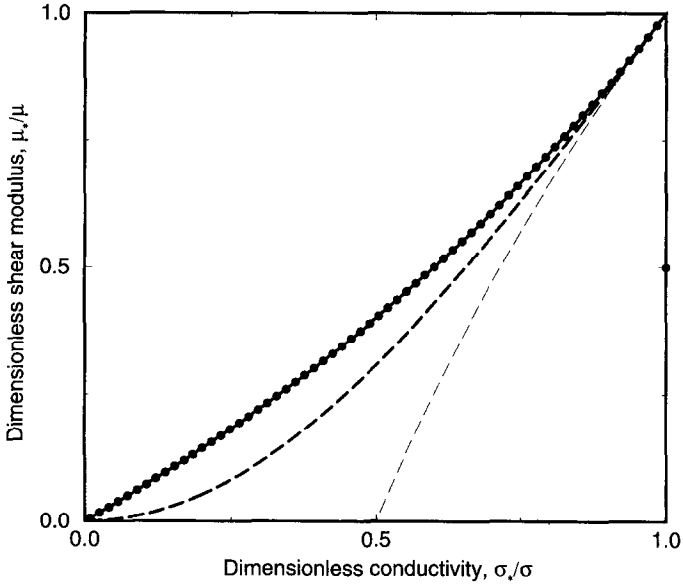

Fig. 2. Comparison of cross-property shear modulus upper bound (2.4), that is valid for arbitrary shape and spatial distribution of cracks, to shear modulus approximations for two-dimensional composites with randomly distributed linear cracks. Self-consistent scheme (3.1), (3.2) (light dashed curve), differential scheme (3.4), (3.5) (bold dashed curve), and non-interacting cracks (3.6), (3.7) (dotted curve). Note that the bound (2.4) (bold solid curve) coincides with the non-interacting cracks approximation (3.6), (3.7) (dotted curve) resulting in a bold solid curve with black dots. The lower bound is trivial and equal to zero.

Figure 2 illustrates the shear modulus bound (2.4) and the aforementioned approximations for a two-dimensional cracked materials. The matrix material has a Poisson's ratio $\nu = 1/3$. Note that the non-interacting cracks approximation (3.6), (3.7) (the dotted curve) coincides with the bound (2.4) (the bold curve) producing the bold curve with the black dots on it. Figure 3 illustrates the bulk modulus bound (2.6), Milton bound (2.9), and the non-interacting cracks approximation (3.8) for a three-dimensional cracked material with a matrix Poisson's ratio $\nu = 1/3$.

Among all of the approximation schemes, the non-interacting cracks approximation lies closest to our upper bound. Moreover, in two dimensions, this approximation for the effective shear modulus coincides with our upper shear modulus bound (2.4). Interestingly, numerical experiments of Kachanov (1992) show that the non-interacting cracks approximation is in good agreement with simulations even for high crack density. The differential scheme approximation lies below the non-interacting cracks curve. The self-consistent scheme gives the lowest values for the effective elastic moduli for a given value of the effective conductivity.

4. CONCLUSIONS

In this paper we obtained the first non-trivial bounds (2.3)–(2.6) on the effective moduli of a cracked material in which the cracks have arbitrary shapes and spatial

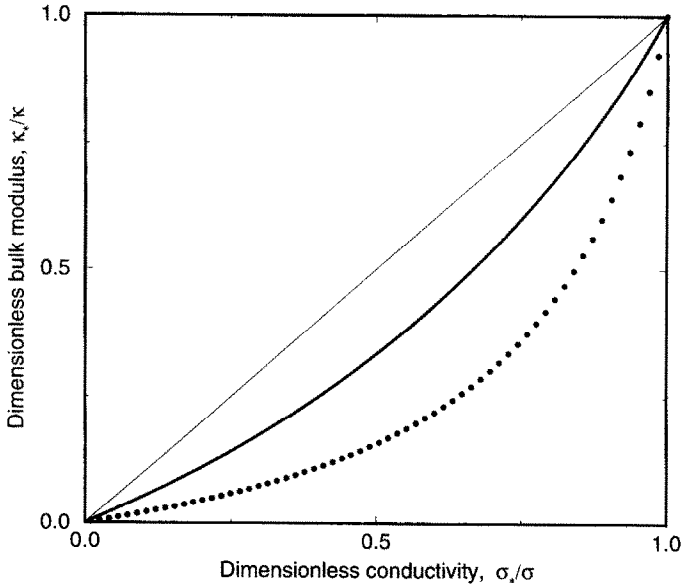

Fig. 3. Comparison of cross-property bulk modulus bound (2.6) (bold curve) and Milton bound (2.9) (the light straight line), that are valid for arbitrary shape and spatial distribution of cracks, to non-interacting cracks approximation (3.6), (3.7) (light dashed curve) for three-dimensional composites with randomly distributed penny-shaped cracks. The lower bound is trivial and equal to zero.

distribution. Specifically, we found an upper bound on the effective elastic moduli of the cracked body in terms of the effective conductivity of this material or, equivalently, lower bound on the effective conductivity of the cracked body in terms of the effective elastic moduli. We also found the differential form of these bounds (2.12)–(2.15). We determined that common approximations for the effective moduli of cracked bodies, i.e. self-consistent scheme, differential scheme, and non-interacting cracks approximations, do not violate the bounds. Moreover, the non-interacting cracks approximation for the shear modulus of a material with randomly distributed linear cracks exactly coincides with our shear modulus bound.

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