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GEOMETRICAL-PARAMETER BOUNDS ON THE EFFECTIVE MODULI OF COMPOSITES

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ABSTRACT

We study bounds on the effective conductivity and elastic moduli of two-phase isotropic composites that depend on geometrical parameters that take into account up to three-point statistical information concerning the composite microstructure. We summarize existing bounds, apply a special fractional linear transformation to simplify their functional forms, and describe two approaches to improve such bounds. These approaches allow us to get new and improved geometrical-parameter bounds on the elastic moduli of two-dimensional composites. Applications of the bounds for effective-medium geometries as well as random arrays of aligned fibers in a matrix are discussed.

1. GEOMETRICAL PARAMETERS AND BOUNDS ON THE EFFECTIVE MODULI

It is well known that effective properties of random two-phase composite materials generally depend upon an infinite set of correlation functions that statistically characterize the microstructure (see review by Torquato (1991) for references). An example of such a correlation function is the so-called n-point probability function S_n defined by the relation

$$S_n(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \left\langle \prod_{i=1}^{n} I(\mathbf{x}_i) \right\rangle = \langle I(\mathbf{x}_1) I(\mathbf{x}_2) \ldots I(\mathbf{x}_n) \rangle, \tag{1.1}$$

where $I(\mathbf{x})$ is the characteristic function of one of the phases, say phase 1, i.e.

$$I(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \text{phase } 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The angular brackets in (1.1) denote an ensemble average. For statistically homogeneous media and under the ergodic hypothesis, one can equate ensemble and volume averages. In particular, the one-point probability function S_1 is the probability of finding a point in phase 1, which is equal to f_1, the volume fraction of phase 1, i.e.

$$S_1 = f_1 = 1 - f_2 = \langle I(\mathbf{x}) \rangle. \tag{1.3}$$

For statistically isotropic media, the quantity $S_2(r)$ is the probability of finding the end points of a line of length r in phase 1 when randomly thrown into the sample. Similarly, for such a materials, $S_3(r, s, t)$ is the probability of finding the vertices of a triangle, with sides of lengths r, s, and t, in phase 1. In general, the infinite set S_1, $S_2, \ldots, S_n$ ($n \to \infty$) is never known and hence an exact determination of the effective properties is not possible. Indeed, in practice, only the first few correlation functions (e.g. S_1, S_2, S_3, and S_4) can be ascertained theoretically for models of composite media (see, e.g. Torquato and Stell (1982), Torquato (1991), and references therein) or experimentally for real materials (Berryman and Blair, 1986).

Given limited microstructural information, the only rigorous statement that can be made about the effective properties must be in the form of inequalities, i.e. rigorous property bounds. In the case of the conductivity (or dielectric constant, magnetic permeability, diffusion coefficient, etc.) and elastic moduli (the subject of this paper), the most well known results are the Hashin–Shtrikman (1962, 1963) bounds which incorporate volume-fraction information only. These bounds, for isotropic composites, actually depend upon the end points of the two-point function S_2, i.e. $S_2(0) = f_1$ and $S_2(\infty) = f_1^2$.

Prager (1963) was the first to derive bounds on the effective diffusion coefficient associated with flow past fixed obstacles, that incorporate the three-point function S_3. This is just the infinite-contrast limit of the conductivity problem. Beran (1965) later obtained bounds on the effective conductivity of three-dimensional isotropic media that also involved the three-point probability function S_3. The Beran bounds were independently shown by Milton (1981a) and by Torquato (1980) to depend upon a single key multidimensional integral, namely,

$$\zeta_1 = 1 - \zeta_2 = \frac{9}{2f_1f_2}\int_0^\infty \frac{\mathrm{d}r}{r}\int_0^\infty \frac{\mathrm{d}s}{s}\int_{-1}^{1} d(\cos\theta)P_2(\cos\theta)\left[S_3(r,s,t) - \frac{S_2(r)S_2(s)}{f_1}\right]. \quad (1.4)$$

Here P_n is a Legendre polynomial of order n and θ is the angle opposite the side of the triangle of length t. The two-dimensional analog of the Beran bounds was obtained by Silnutzer (1972) and was shown by Milton (1982) and Schulgasser (1976b) to involve the parameter

$$\zeta_1 = 1 - \zeta_2 = \frac{4}{\pi f_1 f_2}\int_0^\infty \frac{\mathrm{d}r}{r}\int_0^\infty \frac{\mathrm{d}s}{s}\int_0^\pi d\theta\cos(2\theta)\left[S_3(r,s,t) - \frac{S_2(r)S_2(s)}{f_1}\right]. \quad (1.5)$$

Interestingly, Milton (1981a, 1982) also demonstrated that the parameter ζ_1 arises in bounds on the effective bulk modulus of two-phase isotropic composites due to Beran and Molyneux (1966) for three dimensions and to Silnutzer (1972) for two dimensions.

Three-point bounds on the effective shear modulus (Milton, 1981a, 1982) depend not only on the parameter ζ_1 but on another parameter η_1. In three dimensions it is given by

$$\eta_1 = 1 - \eta_2 = \frac{225}{8f_1f_2}\int_0^\infty \frac{\mathrm{d}r}{r}\int_0^\infty \frac{\mathrm{d}s}{s}\int_{-1}^{1} d(\cos\theta)P_4(\cos\theta)\left[S_3(r,s,t) - \frac{S_2(r)S_2(s)}{f_1}\right], \quad (1.6)$$

and in two dimensions it is given by

$$\eta_1 = 1-\eta_2 = \frac{16}{\pi f_1 f_2}\int_0^\infty \frac{\mathrm{d}r}{r}\int_0^\infty \frac{\mathrm{d}s}{s}\int_0^\pi d\theta \cos(4\theta)\left[S_3(r,s,t)-\frac{S_2(r)S_2(s)}{f_1}\right]. \quad (1.7)$$

The parameter η_1 for $d=3$, (1.6), arises in the Milton–Phan Thien (1982) bounds, as well as the somewhat weaker McCoy (1970) and Quintanilla and Torquato (1995) bounds. The two-dimensional parameter η_1 of (1.7) arises in the Silnutzer bounds (1972). Note that the parameters η_1 and η_2 defined here for $d=3$ are different from the original definitions of Milton and Phan Thien (1982), referred to here as η'_1 and η'_2, respectively. Following a suggestion by Milton (1993), we define the new ones according to (1.6) because such η-parameters are independent of the ζ-parameters. The new parameters η_1, η_2 are related to the old ones η'_1, η'_2 by the expressions

$$\eta_1 = (21\eta'_1 - 5\zeta_1)/16, \quad \eta_2 = (21\eta'_2 - 5\zeta_2)/16. \quad (1.8)$$

It is important to emphasize that all of the aforementioned geometrical parameters lie in the interval [0, 1]. Therefore, the triple (f_1, ζ_1, η_1) belongs to the unit cube. There are no known bounds that would allow one to narrow this cubical region of admissible values without additional microstructural information. Interestingly, as pointed out by Torquato (1991), for an important and common class of composites consisting of inclusions (phase 2) in a matrix (phase 1), the following relations are valid:

$$\zeta_2 \in [0, f_2], \quad \eta_2 \in [\zeta_2, f_2]. \quad (1.9)$$

The geometrical parameters are known analytically for a few structures. For example, assemblages of coated spheres or coated circles that saturate the Hashin–Shtrikman (1962, 1963) bounds on the effective conductivity and bulk modulus (cf. (2.1) and (2.2)) realize the extreme limits of the ζ-parameters, i.e. $\zeta_1 = 1$, when phase 1 is the continuous matrix and phase 2 forms the included phase, and $\zeta_1 = 0$, when phase 1 is the included phase and phase 2 forms the continuous matrix. For the structures that realize the Hashin–Shtrikman bounds on the effective shear modulus (see Lurie and Cherkaev (1985), Norris (1985), Milton (1986), Francfort and Murat (1986)), $\zeta_1 = \eta_1 = 1$, when phase 1 is the continuous matrix and phase 2 forms the included phase, and $\zeta_1 = \eta_1 = 0$ in the opposite situation. This was first noted by Milton (1981a, 1984). Milton (1981a, 1984) also found that for the geometries that correspond to the effective-medium theory, the geometrical parameters are equal to the volume fraction, i.e. $\zeta_1 = \eta_1 = f_1$. The parameter ζ_1 has been determined for some isotropic laminate structures by Schulgasser (1977) and Milton (1981b).

Both ζ_1 and η_1 have been evaluated for certain periodic as well as random arrays of infinitely long, parallel, circular cylinders (circular disks in two dimensions) and random arrays of spheres as a function of the volume fraction (see review of Torquato (1991) for specific references). Recently, the parameter η_1 has been computed for hexagonal arrays of cylinders by Eischen and Torquato (1993) and by Helsing (1994a). The parameter ζ_1 has been calculated for the three-dimensional random checkerboard model by Helsing (1994b).

Property bounds that depend upon the geometrical parameters ζ_1 and η_1 have been shown to provide significant improvement over the Hashin–Shtrikman bounds for moderate phase-contrast ratios. They allow accurate prediction of the effective properties of actual composite materials (see, for example, Davis (1991), Torquato (1991), Davis *et al.*, 1992). Moreover, it is now well established that even when such improved bounds diverge from each other at infinite-contrast ratios, one of the bounds can provide good estimates of the effective properties, depending upon whether the system is below or above the percolation point (see, for example, Torquato, 1985, 1991). One of the main aims of this paper is to improve upon existing geometrical-parameter bounds.

In Section 2 we summarize existing bounds on the effective conductivity and elastic moduli of isotropic two-phase composites that depend on the aforementioned geometrical parameters. By applying the so-called Y-transformation, we are able to express existing bounds in a new and convenient form. In Section 3 we use the translation method originated by Lurie and Cherkaev (1984, 1986) and Murat and Tartar (1985) to improve upon the geometrical-parameter bounds on the elastic moduli of two-dimensional composites. In Section 4 we employ the cross-property conductivity-elastic moduli bounds derived by Gibiansky and Torquato (1993, 1994) to improve upon geometrical-parameter bounds. In Section 5 we discuss the results and apply the bounds to specific geometries.

2. SIMPLIFICATION OF EXISTING BOUNDS

In this section we summarize existing geometrical-parameter bounds and introduce the fractional linear Y-transformation. Then we apply this transformation to simplify the form of the bounds.

2.1. *Hashin–Shtrikman–Walpole bounds on the effective moduli and the Y-transformation*

First let us recall the well-known Hashin–Shtrikman (1962, 1963) bounds on conductivity and bulk modulus of an isotropic composite and the Hashin–Shtrikman (1963) and Walpole (1966) bounds on the effective shear modulus. All of these bounds involve only the volume fractions and can be written in the form

$$F(\sigma_1,\sigma_2,f_1,f_2,(d-1)\sigma_{\min}) \leqslant \sigma_* \leqslant F(\sigma_1,\sigma_2,f_1,f_2,(d-1)\sigma_{\max}), \tag{2.1}$$

$$F\left(\kappa_1,\kappa_2,f_1,f_2,\frac{2(d-1)}{d}\mu_{\min}\right) \leqslant \kappa_* \leqslant F\left(\kappa_1,\kappa_2,f_1,f_2,\frac{2(d-1)}{d}\mu_{\max}\right), \tag{2.2}$$

$$F\left(\mu_1,\mu_2,f_1,f_2,\frac{\kappa_{\min}\mu_{\min}}{\kappa_{\min}+2\mu_{\min}}\right) \leqslant \mu_* \leqslant F\left(\mu_1,\mu_2,f_1,f_2,\frac{\kappa_{\max}\mu_{\max}}{\kappa_{\max}+2\mu_{\max}}\right), \quad \text{if } d=2, \tag{2.3}$$

$$F\left(\mu_1,\mu_2,f_1,f_2,\frac{\mu_{\min}(9\kappa_{\min}+8\mu_{\min})}{6\kappa_{\min}+12\mu_{\min}}\right) \leqslant \mu_* \leqslant F\left(\mu_1,\mu_2,f_1,f_2,\frac{\mu_{\max}(9\kappa_{\max}+8\mu_{\max})}{6\kappa_{\max}+12\mu_{\max}}\right) \quad \text{if} \quad d=3. \tag{2.4}$$

(See also papers by Hashin (1965) and Hill (1964) concerning the two-dimensional elasticity problem). Here σ_*, κ_*, and μ_* are the effective conductivity, bulk and shear moduli, respectively, $d = (2 \text{ or } 3)$ is the space dimension, and subindices min and max denote the minimal and maximal phase moduli, respectively. Moreover, F is the following function of its five variables:

$$F(a_1, a_2, f_1, f_2, y) = f_1 a_1 + f_2 a_2 - \frac{f_1 f_2 (a_1 - a_2)^2}{f_2 a_1 + f_1 a_2 + y}, \tag{2.5}$$

where a represents any phase property. Let us now introduce the so-called Y-transformation (see Cherkaev and Gibiansky, 1992, and Milton, 1991) which is an inverse to the function F as a function of its fifth variable y, i.e.

$$y(a_1, a_2, f_1, f_2, a_*) = -f_2 a_1 - f_1 a_2 + \frac{f_1 f_2 (a_1 - a_2)^2}{f_1 a_1 + f_2 a_2 - a_*}. \tag{2.6}$$

For brevity we sometimes will omit the first four arguments of this function and write it as $y_a(a_*) = y(a_1, a_2, f_1, f_2, a_*)$. One can easily check that the bounds

$$F(a_1, a_2, f_1, f_2, y_1) \leqslant a_* \leqslant F(a_1, a_2, f_1, f_2, y_2), \tag{2.7}$$

are equivalent to the following bounds in terms of the Y-transformation:

$$y_1 \leqslant y_a(a_*) \leqslant y_2 \tag{2.8}$$

Therefore, inequalities (2.1)–(2.4) can be rewritten in the form

$$(d-1)\sigma_{\min} \leqslant y_\sigma(\sigma_*) \leqslant (d-1)\sigma_{\max}, \tag{2.9}$$

$$\frac{2(d-1)}{d}\mu_{\min} \leqslant y_\kappa(\kappa_*) \leqslant \frac{2(d-1)}{d}\mu_{\max}, \tag{2.10}$$

$$\frac{\kappa_{\min}\mu_{\min}}{\kappa_{\min} + 2\mu_{\min}} \leqslant y_\mu(\mu_*) \leqslant \frac{\kappa_{\max}\mu_{\max}}{\kappa_{\max} + 2\mu_{\max}}, \quad \text{if } d = 2, \tag{2.11}$$

$$\frac{\mu_{\min}(9\kappa_{\min} + 8\mu_{\min})}{6\kappa_{\min} + 12\mu_{\min}} \leqslant y_\mu(\mu_*) \leqslant \frac{\mu_{\max}(9\kappa_{\max} + 8\mu_{\max})}{6\kappa_{\max} + 12\mu_{\max}}, \quad \text{if } d = 3. \tag{2.12}$$

It is seen that the Y-transformation allows one to represent the bounds (2.1)–(2.4) in the form (2.9)–(2.12) in that they do not depend explicitly on the phase volume fractions. However, the bounds do depend on the volume fractions implicitly through the definition of the Y-transformation.

Note that simplest bounds on the effective moduli are given by harmonic and arithmetic averages of the phase moduli, i.e.

$$a_h \leqslant a_* \leqslant a_a, \quad \text{where } a_h = \left[\frac{f_1}{a_1} + \frac{f_2}{a_2}\right]^{-1}, \quad a_a = f_1 a_1 + f_2 a_2, \tag{2.13}$$

where a_i are any of the moduli σ_i, κ_i or μ_i. These bounds can be rewritten in the form

$$F(a_1, a_2, f_1, f_2, 0) \leqslant a_* \leqslant F(a_1, a_2, f_1, f_2, \infty), \tag{2.14}$$

or, by use of the Y-transformation, are expressible as

$$0 \leqslant y_a(a_*) \leqslant \infty. \tag{2.15}$$

We will refer to this representation later in the text. Inequalities (2.13) are referred to as the Reuss–Voigt bounds.

2.2. *Beran-type conductivity bounds*

The Beran-type bounds on the effective conductivity σ_* that incorporate volume fractions f_1, f_2 and geometrical parameters ζ_1, ζ_2 (see, e.g. the review of Torquato (1991) for references) can be written in the form

$$F\left(\sigma_1, \sigma_2, f_1, f_2, (d-1)\left[\frac{\zeta_1}{\sigma_1}+\frac{\zeta_2}{\sigma_2}\right]^{-1}\right) \leqslant \sigma_* \leqslant F(\sigma_1, \sigma_2, f_1, f_2, (d-1)[\zeta_1\sigma_1+\zeta_2\sigma_2]). \tag{2.16}$$

By using the Y-transformation we can rewrite (2.16) as follows:

$$(d-1)\left[\frac{\zeta_1}{\sigma_1}+\frac{\zeta_2}{\sigma_2}\right]^{-1} \leqslant y_\sigma(\sigma_*) \leqslant (d-1)[\zeta_1\sigma_1+\zeta_2\sigma_2] \tag{2.17}$$

It is seen that the Y-transformation again allows us to simplify the form of the bounds and "hide" the dependence of the bounds on the volume fractions.

Now our aim is to simplify (2.17) even further. Note that the inequalities (2.17) are similar to the Reuss–Voigt bounds in the form (2.13). The difference is that the volume fractions are replaced by the ζ-parameters, and the phase properties are replaced by the expressions that enter the inequalities (2.9) (i.e. Hashin–Shtrikman bounds in terms of the Y-transformation). Expressions (2.17) can be thus rewritten as

$$F((d-1)\sigma_1, (d-1)\sigma_2, \zeta_1, \zeta_2, 0) \leqslant y_\sigma(\sigma_*) \leqslant F((d-1)\sigma_1, (d-1)\sigma_2, \zeta_1, \zeta_2, \infty) \tag{2.18}$$

or by using again the function y in the form

$$0 \leqslant y((d-1)\sigma_1, (d-1)\sigma_2, \zeta_1, \zeta_2, y_\sigma(\sigma_*)) \leqslant \infty. \tag{2.19}$$

We will call this new transformation the Y_ζ-transformation, although it is based on use of the same function y but with different arguments. In the same manner that the Y-transformation eliminates the explicit dependence of the inequalities (2.1)–(2.4) on the volume fraction, the Y_ζ-transformation eliminates the explicit dependence of the inequalities (2.17) on the geometrical parameters ζ_1, ζ_2.

Beran's three-dimensional lower bound was improved by Milton (1984), who proved that

$$F(2\sigma_1, 2\sigma_2, \zeta_1, \zeta_2, \sigma_{\min}) \leqslant y_\sigma(\sigma_*) \leqslant F(2\sigma_1, 2\sigma_2, \zeta_1, \zeta_2, \infty), \quad \text{for } d=3. \tag{2.20}$$

In two dimensions both upper and lower bounds were improved by Milton (1981b) and are written as

$$F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_{\min}) \leqslant y_\sigma(\sigma_*) \leqslant F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_{\max}), \quad \text{for } d=2. \tag{2.21}$$

The relations (2.21) are surprisingly similar to the expressions (2.1) for $d = 2$. It is natural to rewrite the inequalities (2.20)–(2.21) in the following form

$$\sigma_{\min} \leqslant y(2\sigma_1, 2\sigma_2, \zeta_1, \zeta_2, y_\sigma(\sigma_*)) \leqslant \infty \quad \text{for } d = 3, \tag{2.22}$$

$$\sigma_{\min} \leqslant y(\sigma_1, \sigma_2, \zeta_1, \zeta_2, y_\sigma(\sigma_*)) \leqslant \sigma_{\max} \quad \text{for } d = 2, \tag{2.23}$$

by using the Y_ζ-transformation of the Y-transformations of the effective moduli. Note that in the form (2.22) and (2.23), the bounds do not depend explicitly on either the volume fractions f_1, f_2 (that were eliminated by the Y-transformation) or on the geometrical parameters ζ_1, ζ_2 (that were eliminated by the Y_ζ-transformation).

2.3. *Beran-type elasticity bounds*

Now we turn our attention to the geometrical-parameter bounds for the elasticity problem. The Beran-type bulk modulus bounds in the form obtained by Milton (1981a, 1982) can be written in the form

$$F\left(\frac{2(d-1)\mu_1}{d}, \frac{2(d-1)\mu_2}{d}, \zeta_1, \zeta_2, 0\right) \leqslant y_\kappa(\kappa_*)$$
$$\leqslant F\left(\frac{2(d-1)\mu_1}{d}, \frac{2(d-1)\mu_2}{d}, \zeta_1, \zeta_2, \infty\right) \tag{2.24}$$

or, equivalently,

$$0 \leqslant y\left(\frac{2(d-1)\mu_1}{d}, \frac{2(d-1)\mu_2}{d}, \zeta_1, \zeta_2, y_\kappa(\kappa_*)\right) \leqslant \infty. \tag{2.25}$$

The bounds on the effective shear modulus of a three-dimensional composite were obtained by McCoy (1970) and improved by Milton and Phan Thien (1982). The latter bounds are expressible as

$$\Xi \leqslant y_\mu(\mu_*) \leqslant \Theta, \quad (d = 3), \tag{2.26}$$

where

$$\Xi = \frac{15\langle\mu^{-1}\rangle_\eta + 48\langle\mu^{-1}\rangle_\zeta + 56\langle\kappa^{-1}\rangle_\zeta}{2\langle\mu^{-1}\rangle_\eta(21\langle\mu^{-1}\rangle_\zeta + 2\langle\kappa^{-1}\rangle_\zeta) + 80\langle\mu^{-1}\rangle_\zeta\langle\kappa^{-1}\rangle_\zeta}, \tag{2.27}$$

$$\Theta = \frac{8\langle\mu\rangle_\eta(7\langle\mu\rangle_\zeta + 6\langle\kappa\rangle_\zeta) + 15\langle\mu\rangle_\zeta\langle\kappa\rangle_\zeta}{80\langle\mu\rangle_\eta + 4\langle\mu\rangle_\zeta + 42\langle\kappa\rangle_\zeta}, \tag{2.28}$$

and

$$\langle a\rangle_\zeta = \zeta_1 a_1 + \zeta_2 a_2, \quad \langle a\rangle_\eta = \eta_1 a_1 + \eta_2 a_2. \tag{2.29}$$

Here we have used the definition (1.6) of the parameters η_1, η_2 instead of the definition (1.8) used by Milton and Phan-Thien (1982).

The shear modulus bounds for $d = 2$ obtained by Silnutzer (1972) can be expressed as

$$\left[2\left\langle\frac{1}{\kappa}\right\rangle_\zeta+\left\langle\frac{1}{\mu}\right\rangle_\eta\right]^{-1}\leqslant y_\mu(\mu_*)\leqslant\frac{2\langle\kappa\rangle_\zeta\langle\mu\rangle^2+\langle\kappa\rangle^2\langle\mu\rangle_\eta}{\langle\kappa+2\mu\rangle^2}\quad(d=2),\tag{2.30}$$

where

$$\langle a\rangle=f_1a_1+f_2a_2.\tag{2.31}$$

The upper bound in (2.30) was improved by Kublanov and Milton (1991) who found that

$$\left[2\left\langle\frac{1}{\kappa}\right\rangle_\zeta+\left\langle\frac{1}{\mu}\right\rangle_\eta\right]^{-1}\leqslant y_\mu(\mu_*)\leqslant\left[\frac{1}{\langle\mu\rangle_\eta}+\frac{2}{\langle\kappa\rangle_\zeta}\right]^{-1}.\tag{2.32}$$

The shear modulus bounds (2.26), (2.32) depend on the parameter η_1 which must lie in the interval $\eta_1\in[0,1]$. It is useful in some situations to exclude this parameter from the bounds by taking the extremum over this parameter in the relations (2.26)–(2.32). One can check that all of these bounds are monotonic functions of the parameter η_1 (recall that $\eta_2=1-\eta_1$). Therefore, the minima of the lower bounds and the maxima of the upper bounds are realizable by the values $\eta_1=0$ or $\eta_1=1$ and the new bounds read as follows:

$$\hat{\Xi}\leqslant y_\mu(\mu_*)\leqslant\hat{\Theta},\quad(d=3),\tag{2.33}$$

where

$$\hat{\Xi}=\frac{15\mu_{\min}^{-1}+48\langle\mu^{-1}\rangle_\zeta+56\langle\kappa^{-1}\rangle_\zeta}{2\mu_{\min}^{-1}(21\langle\mu^{-1}\rangle_\zeta+2\langle\kappa^{-1}\rangle_\zeta)+80\langle\mu^{-1}\rangle_\zeta\langle\kappa^{-1}\rangle_\zeta},\tag{2.34}$$

$$\hat{\Theta}=\frac{8\mu_{\max}(7\langle\mu\rangle_\zeta+6\langle\kappa\rangle_\zeta)+15\langle\mu\rangle_\zeta\langle\kappa\rangle_\zeta}{80\mu_{\max}+4\langle\mu\rangle_\zeta+42\langle\kappa\rangle_\zeta},\tag{2.35}$$

and

$$\left[2\left\langle\frac{1}{\kappa}\right\rangle_\zeta+\frac{1}{\mu_{\min}}\right]^{-1}\leqslant y_\mu(\mu_*)\leqslant\left[\frac{1}{\mu_{\max}}+\frac{2}{\langle\kappa\rangle_\zeta}\right]^{-1},\quad(d=2).\tag{2.36}$$

By using the function F we can rewrite the last inequality in the form

$$F\left(\frac{\kappa_1\mu_{\min}}{\kappa_1+2\mu_{\min}},\ \frac{\kappa_2\mu_{\min}}{\kappa_2+2\mu_{\min}},\ \zeta_1,\zeta_2,0\right)\leqslant y_\mu(\mu_*)$$
$$\leqslant F\left(\frac{\kappa_1\mu_{\max}}{\kappa_1+2\mu_{\max}},\ \frac{\kappa_2\mu_{\max}}{\kappa_2+2\mu_{\max}},\ \zeta_1,\zeta_2,-\mu_{\max}\right).\tag{2.37}$$

Indeed, let us compare the lower bounds in the forms (2.36) and (2.37). Both bounds are fractional linear functions of the parameter ζ_1. Both bounds are equal to $\kappa_1\mu_{\min}/(\kappa_1+2\mu_{\min})$ when $\zeta_1=1$, equal to $\kappa_2\mu_{\min}/(\kappa_2+2\mu_{\min})$ when $\zeta_1=0$, and equal to zero when $\zeta_1=\infty$. Fractional linear functions that are equal at three points are identically equal. The same can be checked for the upper bounds in (2.36) and (2.37).

The bounds (2.33) cannot be presented in a similar form (without explicit dependence on the parameters ζ_1, ζ_2) by using the function F. Indeed, they depend on these parameters in a more complicated manner.

Unlike the conductivity and bulk modulus bounds, inequalities (2.37) cannot be simplified simultaneously by using the Y_ζ-transformation. Indeed, the first and second arguments of the function F are different on the right- and left-hand sides of (2.37). Therefore, we cannot simplify both the lower and upper bounds by using the same transformation. However, the following transformations may simplify these bounds separately, i.e.

$$0 \leqslant y\left(\frac{\kappa_1\mu_{\min}}{\kappa_1+2\mu_{\min}},\ \frac{\kappa_2\mu_{\min}}{\kappa_2+2\mu_{\min}},\ \zeta_1,\zeta_2,y_\mu(\mu_*)\right) \tag{2.38}$$

and

$$y\left(\frac{\kappa_1\mu_{\max}}{\kappa_1+2\mu_{\max}},\ \frac{\kappa_2\mu_{\max}}{\kappa_2+2\mu_{\max}},\ \zeta_1,\zeta_2,y_\mu(\mu_*)\right) \leqslant -\mu_{\max}. \tag{2.39}$$

To summarize, we have presented the geometrical-parameter bounds in a more compact form by using Y- and Y_ζ-transformations. Now our goal is to improve these bounds.

3. TRANSLATION METHOD AND IMPROVEMENT OF THE GEOMETRICAL-PARAMETER BOUNDS

In this section we use the translation method to improve the geometrical-parameter bounds on the effective moduli of two-dimensional elastic composite. We describe the procedure in Section 3.1 and apply it to the plane-elasticity problem in Section 3.2.

3.1. *Translation method applied to geometrical-parameter bounds*

Here we describe a simple method that one can employ to judge the quality of elastic moduli bounds. This test is useful in the following sense: all "good" bounds have to satisfy the conditions of this test, and all "bad" bounds are improved by this procedure in order to satisfy the test. The procedure is based on the so-called translation method that was introduced independently by Lurie and Cherkaev (1984, 1985), and by Murat and Tartar (1985) and Tartar (1985). We briefly outline the translation method in order to make the paper self-contained. Further details and historical references can be found in the papers by Milton (1990), Cherkaev and Gibiansky (1993), and Gibiansky and Torquato (1994).

Consider a two-phase composite with the local constitutive relation

$$\mathbf{j}(\mathbf{x}) = \mathbf{D}(\mathbf{x})\cdot\mathbf{e}(\mathbf{x}) \tag{3.1}$$

at a point $\mathbf{x}$. Here $\mathbf{j}$ is a generalized "flux", $\mathbf{e}$ is a generalized "gradient", and $\mathbf{D}$ is some local property, generally a tensor, equal to $\mathbf{D}_1$ in phase 1 and $\mathbf{D}_2$ in phase 2. For example, in the conduction (elasticity) problem, $\mathbf{j}$, $\mathbf{e}$ and $\mathbf{D}$ represent the current (stress), electric field (strain), and conductivity tensor (stiffness tensor), respectively. The effective tensor $\mathbf{D}_*$ can be defined by the variational principle

$$\mathbf{e}_0 \cdot \mathbf{D}_* \cdot \mathbf{e}_0 = \inf_{\substack{\mathbf{e}:\langle \mathbf{e}\rangle = \mathbf{e}_0, \\ \mathbf{e}\in\mathscr{E}}} \langle \mathbf{e}\cdot\mathbf{D}(\mathbf{x})\cdot\mathbf{e}\rangle, \tag{3.2}$$

where $\mathscr{E}$ is the set of the field $\mathbf{e}(\mathbf{x})$ that satisfy some differential restrictions depending on the physical problem. For example, any field $\mathbf{e}$ that belongs to the set $\mathscr{E}$ of electrical fields should be expressible as the gradient of a potential, $\mathbf{e} = \nabla\phi$, or any stress field τ should satisfy the equilibrium conditions $\nabla \cdot \tau = 0$.

Now consider a "comparison" medium with local property tensor

$$\mathbf{D}'(\mathbf{x}) = \mathbf{D}(\mathbf{x}) - \mathbf{T}, \tag{3.3}$$

where $\mathbf{T}$ is a constant translation tensor chosen in such a way that:

(i) $\mathbf{D}' = \mathbf{D}(\mathbf{x}) - \mathbf{T}$ is positive semi-definite and
(ii) the quadratic form associated with $\mathbf{T}$ is quasiconvex, i.e. such that

$$\langle \mathbf{e}\cdot\mathbf{T}\cdot\mathbf{e}\rangle \geqslant \langle \mathbf{e}\rangle\cdot\mathbf{T}\cdot\langle \mathbf{e}\rangle \text{ for any } \mathbf{e}\in\mathscr{E}. \tag{3.4}$$

The effective properties of such a medium can be defined via

$$\mathbf{e}_0 \cdot \mathbf{D}'_* \cdot \mathbf{e}_0 = \inf_{\substack{\mathbf{e}:\langle \mathbf{e}\rangle = \mathbf{e}_0, \\ \mathbf{e}\in\mathscr{E}}} \langle \mathbf{e}\cdot(\mathbf{D}(\mathbf{x}) - \mathbf{T})\cdot\mathbf{e}\rangle \tag{3.5}$$

(cf. (3.2)). Let $\mathbf{e}'(\mathbf{x})$ be a solution of the variational problem (3.2) and let us use this field as a trial field for the variational problem (3.5). This yields

$$\mathbf{e}_0 \cdot \mathbf{D}'_* \cdot \mathbf{e}_0 \leqslant \langle \mathbf{e}'\cdot\mathbf{D}(\mathbf{x})\cdot\mathbf{e}'\rangle - \langle \mathbf{e}'\cdot\mathbf{T}\cdot\mathbf{e}'\rangle \leqslant \mathbf{e}_0\cdot\mathbf{D}_*\cdot\mathbf{e}_0 - \mathbf{e}_0\cdot\mathbf{T}\cdot\mathbf{e}_0, \tag{3.6}$$

where we took into account of the quasiconvexity of the quadratic form with the matrix $\mathbf{T}$ and (3.2) that is an equality for the field $\mathbf{e} = \mathbf{e}'$. Hence, the effective properties of the comparison and original media are related by

$$\mathbf{D}_* - \mathbf{T} \geqslant \mathbf{D}'_*. \tag{3.7}$$

Now the usual procedure of the translation method assumes the use of the well-known harmonic-mean bound that yields

$$(\mathbf{D}_* - \mathbf{T}) \geqslant \mathbf{D}'_* \geqslant [f_1(\mathbf{D}_1 - \mathbf{T})^{-1} + (\mathbf{D}_2 - \mathbf{T})^{-1}]^{-1}, \tag{3.8}$$

or

$$(\mathbf{D}_* - \mathbf{T})^{-1} \leqslant f_1(\mathbf{D}_1 - \mathbf{T})^{-1} + f_2(\mathbf{D}_2 - \mathbf{T})^{-1} \tag{3.9}$$

that is true for any matrix $\mathbf{T}$ of a quasiconvex quadratic form such that

$$\mathbf{D}(\mathbf{x}) - \mathbf{T} \geqslant 0 \quad \text{for any } \mathbf{x}. \tag{3.10}$$

For two-phase composites, the restriction (3.10) means

$$\mathbf{D}_1 - \mathbf{T} \geqslant 0, \quad \mathbf{D}_2 - \mathbf{T} \geqslant 0. \tag{3.11}$$

The essential point is that one wants to choose $\mathbf{T}$ so as to optimize the bound, i.e. to make it as restrictive as possible for the effective property tensor $\mathbf{D}_*$. One can transform the bounds by using the so-called Y-transformation (Milton, 1991, Cherkaev and Gibiansky, 1992):

$$\mathbf{Y}(\mathbf{D}_1, \mathbf{D}_2, f_1, f_2, \mathbf{D}_*) = -f_2\mathbf{D}_1 - f_1\mathbf{D}_2 - f_1 f_2(\mathbf{D}_1 - \mathbf{D}_2) \cdot (\mathbf{D}_* - f_1\mathbf{D}_1 - f_2\mathbf{D}_2)^{-1} \cdot (\mathbf{D}_1 - \mathbf{D}_2). \quad (3.12)$$

The scalar form of this transformation was given by (2.6) and was used in this paper to simplify the form of the bounds. It will be convenient at times to omit the first four arguments of the Y-transformation and denote it simply as $\mathbf{Y}(\mathbf{D}_*)$. Through this transformation, the bound (3.9) can be presented in the following surprisingly simple form (Milton, 1991, Cherkaev and Gibiansky, 1992):

$$\mathbf{Y}(\mathbf{D}_*) + \mathbf{T} \geqslant 0. \quad (3.13)$$

The idea of Milton (1993) that we use here is to replace (in the translation method) the harmonic-mean bound on the effective tensor $\mathbf{D}'_*$ of the "comparison" medium by more sophisticated bounds, such as for example, the Beran-type geometrical-parameter bounds. A similar idea was used by Helsing (1993) who combined the Hashin–Shtrikman method and the translation method to get improved bounds on the effective conductivity of a random conducting polycrystal. Let us assume that instead of the bound (3.8) we have some other bound on the effective tensor $\mathbf{D}_*$ in the form

$$\mathbf{D}_* - \mathbf{\Phi}(\mathbf{D}_1, \mathbf{D}_2, I(\mathbf{x})) \geqslant 0. \quad (3.14)$$

The symbolic notation $I(\mathbf{x})$ (cf. (1.2)) as an argument of the tensor function $\mathbf{\Phi}$ means that the bound may depend on the microstructural parameters like volume fraction or the geometrical parameters ζ_1 and η_1. For the "comparison" material this yields

$$\mathbf{D}'_* - \mathbf{\Phi}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x})) \geqslant 0. \quad (3.15)$$

The combination of the inequalities (3.7) and (3.15) gives the bound

$$\mathbf{D}_* - \mathbf{T} \geqslant \mathbf{\Phi}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x})). \quad (3.16)$$

In terms of Y-transformation it can be rewritten as

$$\mathbf{Y}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, f_1, f_2, \mathbf{D}_* - \mathbf{T}) - \mathbf{\Phi}_{\mathrm{Y}}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x})) \geqslant 0, \quad (3.17)$$

where

$$\mathbf{\Phi}_{\mathrm{Y}}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x})) = \mathbf{Y}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, f_1, f_2, \mathbf{\Phi}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x}))) \quad (3.18)$$

is the Y-transformation of the left-hand side of the inequality bound (3.15). As follows from the definition of the Y-transformation,

$$\mathbf{Y}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, f_1, f_2, \mathbf{D}_* - \mathbf{T}) = \mathbf{Y}(\mathbf{D}_1, \mathbf{D}_2, f_1, f_2, \mathbf{D}_*) + \mathbf{T}. \quad (3.19)$$

This leads to the final form of the bounds that we will use, namely,

$$\mathbf{Y}(\mathbf{D}_*) \geqslant \mathbf{\Phi}_{\mathrm{Y}}(\mathbf{D}_1 - \mathbf{T}, \mathbf{D}_2 - \mathbf{T}, I(\mathbf{x})) - \mathbf{T}. \quad (3.20)$$

Again, this inequality is valid for any matrix $\mathbf{T}$ of quasiconvex quadratic form. Usually the derivation of the bound (3.14) requires the condition $\mathbf{D}(\mathbf{x}) \geqslant 0$. This leads to the restriction (3.11) on the translation matrix $\mathbf{T}$. However, one has to be careful to check whether some other assumptions (e.g. the positiveness of the Poisson's ratio of the phases or the proportionality between the phase's moduli) were used in the derivation of the bound (3.14). The translation by the matrix $\mathbf{T}$ should not violate any such assumptions.

Let us now adopt the aforementioned general scheme to improve the bounds on the bulk and shear moduli of two-dimensional elastic composites. The local constitutive relation that replaces (3.1) is

$$\boldsymbol{\tau}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}). \tag{3.21}$$

Here $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau}$ are the stress and strain tensors, respectively, and $\mathbf{C}(\mathbf{x})$ is a stiffness matrix that is equal to $\mathbf{C}_1$ in phase 1 and $\mathbf{C}_2$ in phase 2. The stiffness tensor of an isotropic body is defined by two bulk and shear moduli, i.e. $\mathbf{C}_1 = \mathbf{C}(\kappa_1, \mu_1)$, $\mathbf{C}_2 = \mathbf{C}(\kappa_2, \mu_2)$, and $\mathbf{C}_* = \mathbf{C}(\kappa_*, \mu_*)$.

It is known (see, e.g. Cherkaev and Gibiansky, 1993) that the matrix $\mathbf{T} = \mathbf{C}(-t, t)$ (that formally looks like the stiffness matrix with the bulk modulus equal to $-t$ and shear modulus equal to t) is associated with the quasiconvex quadratic form of the stress tensor for any positive values of the parameter $t \geqslant 0$. Therefore, the "comparison" composite is made of the phases with properties $\mathbf{C}_i - \mathbf{T} = \mathbf{C}(\kappa_i + t, \mu_i - t)$.

Let us now assume that one has a lower bound on the effective bulk modulus of the composite in the form (cf. (3.14))

$$\kappa_* - K^1(\kappa_1, \kappa_2, \mu_1, \mu_2, I(\mathbf{x})) \geqslant 0. \tag{3.22}$$

The new bound (3.15) can be written in the form

$$\kappa_* \geqslant K^1(\kappa_1 + t, \kappa_2 + t, \mu_1 - t, \mu_2 - t, I(\mathbf{x})) - t. \tag{3.23}$$

This is valid for any matrix $\mathbf{T}$ of quasiconvex quadratic form that satisfies (3.11) with $t \geqslant 0$, i.e. for any t such that

$$t \in [0, \mu_{\min}]. \tag{3.24}$$

One wants to choose t so as to optimize the bound. The best bounds maximize the right-hand sides of the expression (3.23) over the allowable range of the parameter t, i.e.

$$\kappa_* \geqslant \max_{t \in [0, \mu_{\min}]} \{K^1(\kappa_1 + t, \kappa_2 + t, \mu_1 - t, \mu_2 - t, I(\mathbf{x})) - t\}. \tag{3.25}$$

Note that the functions K^1 may incorporate information about volume fractions, geometrical parameters ζ_i and η_i, or higher-order parameters. The procedure described above can be applied to any bound of this type. Optimal bounds are maximized by the value $t = 0$ in (3.25), i.e. are stable under the described procedure.

The crucial point in this procedure is that the tensor of the properties of the "translated" material still has the same form as that of the original material, i.e. it can be defined by the "translated" bulk and shear moduli. This is not always true. For example, for the conductivity problem there is no translation that lies within the class of the conductivity matrices. Therefore, the described procedure cannot be applied directly to the Beran conductivity bounds.

The Y-transformation turns out to be very successful in simplifying the form of the bounds. It is natural to use the bounds in terms of the Y-transformation to implement the procedure described above. Applying the Y-transformation to the bound (3.25), we get

$$y_\kappa(\kappa_*) \geqslant \max_{t\in[0,\mu_{\min}]} \{K^{\mathrm{l}}_{\mathrm{Y}}(\kappa_1+t,\kappa_2+t,\mu_1-t,\mu_2-t,I(\mathbf{x}))+t\}, \tag{3.26}$$

where $K^{\mathrm{l}}_{\mathrm{Y}}(\kappa_1,\kappa_2,\mu_1,\mu_2,I(\mathbf{x}))$ is the bound on the Y-transformation of the effective bulk modulus. This expression is usually simpler than the corresponding bound on the modulus itself.

Similarly, given the lower bound on the effective shear modulus

$$\mu_* \geqslant M^{\mathrm{l}}(\kappa_1,\kappa_2,\mu_1,\mu_2,I(\mathbf{x})) \tag{3.27}$$

and using the same procedure, one can arrive at the inequality

$$y_\mu(\mu_*) \geqslant \max_{t\in[0,\mu_{\min}]} \{M^{\mathrm{l}}_{\mathrm{Y}}(\kappa_1+t,\kappa_2+t,\mu_1-t,\mu_2-t,I(\mathbf{x}))-t\}, \tag{3.28}$$

where $M^{\mathrm{l}}_{\mathrm{Y}}(\kappa_1,\kappa_2,\mu_1,\mu_2,I(\mathbf{x}))$ is the Y-transformation of the expression $M^{\mathrm{l}}(\kappa_1, \kappa_2, \mu_1, \mu_2, I(\mathbf{x}))$.

A similar procedure applied to the upper bounds $\kappa_*^{-1} \geqslant K^{\mathrm{u}}(\kappa_1^{-1},\kappa_2^{-1},\mu_1^{-1}, \mu_2^{-1},\mathrm{I}(\mathbf{x}))$ and $\mu_*{}^{-1} \geqslant M^{\mathrm{u}}(\kappa_1^{-1},\kappa_2^{-1},\mu_1^{-1},\mu_2^{-1},I(\mathbf{x}))$ leads to the expressions

$$y_{\kappa^{-1}}(\kappa_*{}^{-1}) \geqslant \max_{t\in[-\kappa_{\max}^{-1},\mu_{\max}^{-1}]} \{K^{\mathrm{u}}_{\mathrm{Y}}(\kappa_1^{-1}+t,\kappa_2^{-1}+t,\mu_1^{-1}-t,\mu_2^{-1}-t,I(\mathbf{x}))+t\}, \tag{3.29}$$

$$y_{\mu^{-1}}(\mu_*^{-1}) \geqslant \max_{t\in[-\kappa_{\max}^{-1},\mu_{\max}^{-1}]} \{M^{\mathrm{u}}_{\mathrm{Y}}(\kappa_1^{-1}-t,\kappa_2^{-1}-t,\mu_1^{-1}+t,\mu_2^{-1}+t,I(\mathbf{x}))-t\}, \tag{3.30}$$

where we have used the translation matrix $\mathbf{T} = \mathbf{C}(-t, t)$. The quadratic form of the strain tensor associated with this matrix is quasiconvex for any t (Cherkaev and Gibiansky, 1993).

3.2. *Application of the method to test and improve existing bounds*

To illustrate the procedure we first apply the bounds (3.29) and (3.30) to improve the lower Hashin–Shtrikman bounds on the effective bulk and shear moduli, given just volume-fraction information. We know that these bounds are optimal but this exercise is useful in being able to understand the method. The Hashin–Shtrikman lower bound on the effective bulk modulus leads to the following inequality (cf. (2.10)):

$$y_\kappa(\kappa_*) \geqslant \max_{t\in[0,\mu_{\min}]} \{(\mu_{\min}-t)+t\} = \mu_{\min}. \tag{3.31}$$

This result obviously coincides with the lower Hashin–Shtrikman bound on the bulk modulus. The translation parameter t does not enter the final expression for the bound as it is canceled out in this expression. The same is true for the upper bounds on bulk and shear moduli. The lower Hashin–Shtrikman bound on the shear modulus leads to the inequality

$$y(\mu_*) \geqslant \max_{t\in[0,\mu_{\min}]} \left\{\frac{2(\kappa_{\min}+t)(\mu_{\min}-t)}{(\kappa_{\min}+2\mu_{\min}-t)} - t\right\}. \tag{3.32}$$

One can check that the derivative of the function on the right-hand side of the inequality (3.32) is negative. Therefore, the maximum is attained by the zero value of the parameter t. Therefore, the Hashin–Shtrikman lower bound is also stable under the described procedure, as was expected.

Now we will show how to use this method to improve the Beran-type upper bounds on the bulk and shear moduli for $d = 2$. In terms of the Y-transformations the bulk modulus upper bound (2.24) can be rewritten as follows:

$$y(\kappa_1^{-1}, \kappa_2^{-1}, f_1, f_2, \kappa_*^{-1}) \geqslant \left[\frac{\zeta_1}{\mu_1^{-1}} + \frac{\zeta_2}{\mu_2^{-1}}\right]^{-1}. \tag{3.33}$$

Here we use the following property of the Y-transformation

$$y(\kappa_1^{-1}, \kappa_2^{-1}, f_1, f_2, \kappa_*^{-1}) = y^{-1}(\kappa_1, \kappa_2, f_1, f_2, \kappa_*) \tag{3.34}$$

that can be checked by straightforward calculation. The bound (3.29) where K_Y^u is given by (3.33) leads to the inequality

$$y(\kappa_1^{-1}, \kappa_2^{-1}, f_1, f_2, \kappa_*^{-1}) \geqslant \max_{t\in[-\kappa_{\max}^{-1}, \mu_{\max}^{-1}]} \left\{\left[\frac{\zeta_1}{\mu_1^{-1}-t} + \frac{\zeta_2}{\mu_2^{-1}-t}\right]^{-1} + t\right\}. \tag{3.35}$$

The derivative of the function on the right-hand side of (3.35) is negative, the maximum is attained by the value $t = -\kappa_{\max}^{-1}$ leading to the bound

$$y(\kappa_1^{-1}, \kappa_2^{-1}, f_1, f_2, \kappa_*^{-1}) \geqslant \left\{\left[\frac{\zeta_1}{\mu_1^{-1}+\kappa_{\max}^{-1}} + \frac{\zeta_2}{\mu_2^{-1}+\kappa_{\max}^{-1}}\right]^{-1} - \kappa_{\max}^{-1}\right\}, \tag{3.36}$$

which can be rewritten as

$$y(\kappa_*) \leqslant F(\mu_1, \mu_2, \zeta_1, \zeta_2, \kappa_{\max}), \tag{3.37}$$

or, equivalently, as

$$y(\mu_1, \mu_2, \zeta_1, \zeta_2, y(\kappa_*)) \leqslant \kappa_{\max}. \tag{3.38}$$

The result (3.37) or (3.38) should be compared with the upper Beran bound (2.24) on the effective bulk modulus for $d = 2$. The key point is that (3.37) improves upon (2.24).

Similarly, the Kublanov–Milton upper bound in the form (2.36) (that contains only the parameter ζ_1) can be rewritten in the form

$$y(\mu_1^{-1}, \mu_2^{-1}, f_1, f_2, \mu_*^{-1}) \geqslant \frac{1}{\mu_{\max}} + 2\left\langle\frac{1}{\kappa^{-1}}\right\rangle_\zeta^{-1}. \tag{3.39}$$

The application of the bounds (3.30) with the function M_Y^u given by the right-hand side of the inequality (3.39) yields the result

$$y(\mu_1^{-1},\mu_2^{-1},f_1,f_2,\mu_*^{-1}) \geqslant \max_{t\in[-\kappa_{\max}^{-1},\mu_{\max}^{-1}]}\left\{\frac{1}{\mu_{\max}}-2t+2\left\langle\frac{1}{\kappa^{-1}+t}\right\rangle_\zeta^{-1}\right\}. \tag{3.40}$$

This bound is maximized by the value $t=\mu_{\max}^{-1}$ that gives

$$y_\mu^{-1}(\mu_1,\mu_2,f_1,f_2,\mu_*) = y_\mu^{-1}(\mu_1^{-1},\mu_2^{-1},f_1,f_2,\mu_*^{-1}) \geqslant 2\left\langle\frac{1}{\kappa^{-1}+\mu_{\max}^{-1}}\right\rangle_\zeta^{-1}-\frac{1}{\mu_{\max}} \tag{3.41}$$

or, equivalently,

$$y_\mu(\mu_*) \leqslant F\left(\frac{\kappa_1\mu_{\max}}{\kappa_1+2\mu_{\max}},\frac{\kappa_2\mu_{\max}}{\kappa_2+2\mu_{\max}},\zeta_1,\zeta_2,\mu_{\max}\right). \tag{3.42}$$

We see that upper bound (3.41) or (3.42) improves upon the Kublanov–Milton upper bound (2.36) or (2.37). It is interesting to note that one can get the same result (3.41) by using the Silnutzer upper bound in the form (2.30) instead of the Kublanov and Milton bounds (2.36).

We have improved the shear modulus upper bound in the form (2.36) that does not contain the geometrical parameter η_1. It is interesting to see what the procedure can yield when applied to the sharper bound (2.32). Application of the bound (3.30) yields the inequality

$$y(\mu_1^{-1},\mu_2^{-1},f_1,f_2,\mu_*) \geqslant \max_{t\in[-\kappa_{\max}^{-1},\mu_{\max}^{-1}]}\left\{2\left\langle\frac{1}{\kappa^{-1}+t}\right\rangle_\zeta^{-1}+\left\langle\frac{1}{\mu^{-1}-t}\right\rangle_\eta^{-1}-t\right\}. \tag{3.43}$$

The function on the right-hand side of the inequality (3.43) attains its maximum

$$y_{1*} = \langle 2\kappa^{-1}\rangle_\zeta+\langle\mu^{-1}\rangle_\eta-\frac{\left(\sqrt{2\zeta_1\zeta_2(\kappa_1^{-1}-\kappa_2^{-1})^2}+\sqrt{\eta_1\eta_2(\mu_1^{-1}-\mu_2^{-1})^2}\right)^2}{\eta_1\mu_2^{-1}+\eta_2\mu_1^{-1}+2\zeta_1\kappa_2^{-1}+2\zeta_2\kappa_1^{-1}} \tag{3.44}$$

at the point

$$t=t_*=\frac{\sqrt{2\zeta_1\zeta_2(\kappa_1^{-1}-\kappa_2^{-1})^2}(\eta_1\mu_2^{-1}+\eta_2\mu_1^{-1})-\sqrt{\eta_1\eta_2(\mu_1^{-1}-\mu_2^{-1})^2}(\zeta_1\kappa_2^{-1}+\zeta_2\kappa_1^{-1})}{\sqrt{2\zeta_1\zeta_2(\kappa_1^{-1}-\kappa_2^{-1})^2}+\sqrt{\eta_1\eta_2(\mu_1^{-1}-\mu_2^{-1})^2}}. \tag{3.45}$$

We should, however, check whether the value t_* belongs to the admissible interval (3.24) for the parameter t. If

$$t_* \leqslant -\kappa_{\max}^{-1} \tag{3.46}$$

then the bound is attained by the boundary value $t=-\kappa_{\max}^{-1}$ and is given by

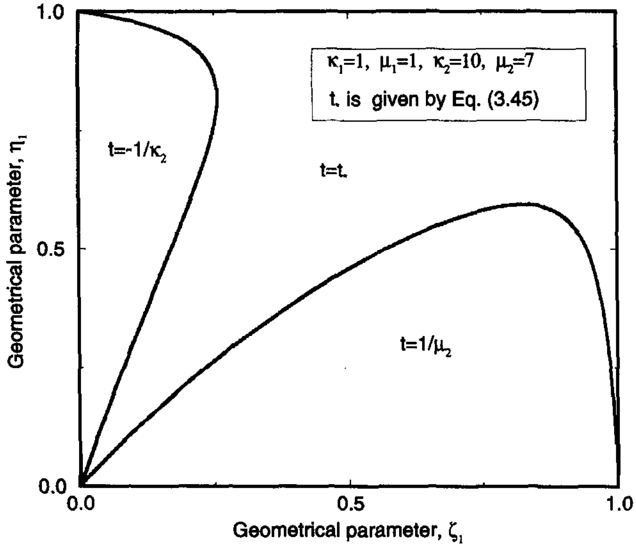

Fig. 1. Regions in the ζ_1–η_1 plane that correspond to optimal values of t.

$$y_{2*} = \left\langle \frac{1}{\mu^{-1} + \kappa_{\max}^{-1}} \right\rangle_\eta^{-1} + \frac{1}{\kappa_{\max}}. \tag{3.47}$$

If

$$t_* \geqslant \mu_{\max}^{-1}, \tag{3.48}$$

then the bound is attained by the boundary value $t = \mu_{\max}^{-1}$ and is given by

$$y_{3*} = 2\left\langle \frac{1}{\kappa^{-1} + \mu_{\max}^{-1}} \right\rangle_\zeta^{-1} - \frac{1}{\mu_{\max}}. \tag{3.49}$$

Summarizing the obtained results and taking into account that $y_\mu^{-1}(\mu_*^{-1}) = y_\mu^{-1}(\mu_*)$, we arrive at the inequality

$$y_\mu(\mu_*) \leqslant A^{-1}, \tag{3.50}$$

where

$$A = \begin{cases} y_{1*}, & \text{if } t_* \in [-\kappa_{\max}^{-1}, \mu_{\max}^{-1}], \\ y_{2*}, & \text{if } t_* \leqslant -\kappa_{\max}^{-1}, \\ y_{3*}, & \text{if } t_* \geqslant \mu_{\max}^{-1}. \end{cases} \tag{3.51}$$

Here t_* is given by the relation (3.45), and y_{1*}, y_{2*}, and y_{3*} are given by (3.44), (3.47), and (3.49), respectively. Figure 1 illustrates the regions in the plane $\zeta_1 - \eta_1$ that correspond to $t_* \in [-\kappa_{\max}^{-1}, \mu_{\max}^{-1}]$, $t_* \leqslant -\kappa_{\max}^{-1}$, and $t_* \geqslant \mu_{\max}^{-1}$, respectively, where we take

$$\kappa_1 = 1, \quad \kappa_2 = 10, \quad \mu_1 = 1, \quad \mu_2 = 7. \tag{3.52}$$

Recall that the pair (ζ_1, η_1) belongs to the unit square in this plane.

Note that if $t_* \geqslant \mu_{\max}^{-1}$, then bound (3.51) (which incorporates the η-parameters) coincides with bound (3.41) (which incorporates only the ζ-parameters).

One can check that both the bulk and shear moduli bounds in three dimensions as well as the lower bound on the shear modulus in two dimensions cannot be improved by the aforementioned method, i.e. the optimal value of the translation parameter t in all these cases is equal to zero.

4. CROSS-PROPERTY BOUNDS AND IMPROVEMENT OF THE GEOMETRICAL-PARAMETER BOUNDS

In this section we describe an alternative method, cross-property bounds, to improve geometrical-parameter bounds. This method is capable of reproducing all of the bounds that were obtained in the previous section by using the translation method except the bound (3.51) that includes the η-parameters. In addition, we are able to get a new lower bound on the shear modulus in two dimensions.

We demonstrate the method on the shear modulus lower bound in two dimensions. First we must collect all of the results that we need for this purpose.

Statement 1: The bounds on the set of the pairs $(y_\sigma(\sigma_*), y_\mu(\mu_*))$ are given in the $y_\sigma(\sigma_*) - y_\mu(\mu_*)$ plane by the outermost of the segments of the hyperbolas

$$\mathrm{Hyp}\,[(\sigma_1, y_1), (\sigma_2, y_2), (-\sigma_1, -\mu_1)], \tag{4.1}$$

$$\mathrm{Hyp}\,[(\sigma_1, y_1), (\sigma_2, y_2), (-\sigma_2, -\mu_2)], \tag{4.2}$$

$$\mathrm{Hyp}\,[(\sigma_1, y_3), (\sigma_2, y_4), (-\sigma_1, -\mu_1)], \tag{4.3}$$

$$\mathrm{Hyp}\,[(\sigma_1, y_3), (\sigma_2, y_4), (-\sigma_2, -\mu_2)], \tag{4.4}$$

in conjunction with the inequalities

$$\sigma_{\min} \leqslant y_\sigma(\sigma_*) \leqslant \sigma_{\max}. \tag{4.5}$$

Here

$$y_1 = \frac{\kappa_1 \mu_{\min}}{\kappa_1 + 2\mu_{\min}}, \quad y_2 = \frac{\kappa_2 \mu_{\min}}{\kappa_2 + 2\mu_{\min}}, \quad y_3 = \frac{\kappa_1 \mu_{\max}}{\kappa_1 + 2\mu_{\max}}, \quad y_4 = \frac{\kappa_2 \mu_{\max}}{\kappa_2 + 2\mu_{\max}}. \tag{4.6}$$

We denote as $\mathrm{Hyp}[(x_1, y_1), (x_2, y_2), (x_3, y_3)]$ the segment of the hyperbola that can be parametrically described as

$$x_* = F(x_1, x_2, \gamma_1, \gamma_2, -x_3), \quad y_* = F(y_1, y_2, \gamma_1, \gamma_2, -y_3), \quad \gamma_1 = 1 - \gamma_2 \in [0, 1]. \tag{4.7}$$

This statement was proved by Gibiansky and Torquato (1993, 1994).

Statement 2: For any fixed value of the parameter ζ_1, the Y-transformation $y_\sigma(\sigma_*)$ of the effective conductivity σ_* is restricted by the inequalities

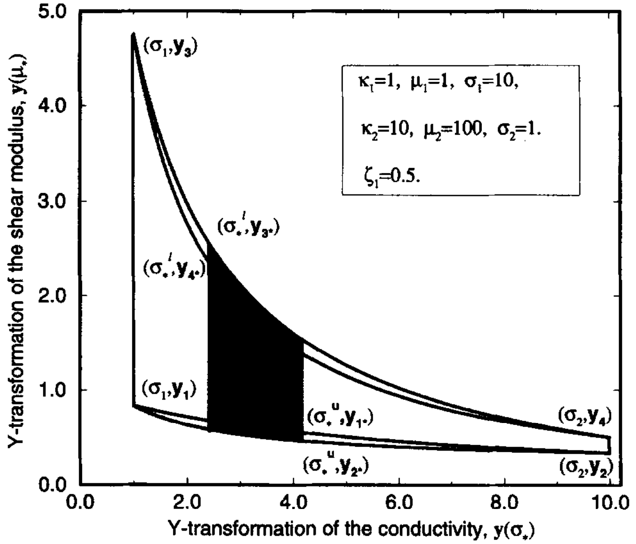

Fig. 2. Construction of the new geometrical-parameter lower bound on the effective shear modulus. Hyperbolas show cross-property bounds of Statement 1, vertical lines show geometrical-parameter conductivity bounds of Statement 2. Pairs $(y(\sigma_*), y(\mu_*))$ in the shaded region satisfy both sets of bounds.

$$\sigma_*^{\mathrm{l}} \leqslant y_\sigma(\sigma_*) \leqslant \sigma_*^{\mathrm{u}}, \tag{4.8}$$

where

$$\sigma_*^{\mathrm{l}} = F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_{\min}), \quad \sigma_*^{\mathrm{u}} = F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_{\max}). \tag{4.9}$$

Statement 2 follows from Milton's bounds (2.21).

Let us now assume that

$$\kappa_1 \geqslant \kappa_2, \tag{4.10}$$

and

$$\sigma_1 \leqslant \sigma_2. \tag{4.11}$$

We emphasize that the elastic moduli are given arbitrarily; condition (4.11) is not a restriction but simply is a labeling of the material components. We treat the conductivity constants as parameters of the problem and choose them in order to get the best bounds out of Statements 1 and 2.

The bounds of Statements 1 and 2 are illustrated in Fig. 2 where the plane $y_\sigma(\sigma_*) - y_\mu(\mu_*)$ is shown. The solid lines correspond to the bounds of Statement 1 and the dashed vertical lines correspond to the bounds (2.21) for fixed ζ_1. It is seen that for fixed ζ_1, the pair $(y_\sigma(\sigma_*), y_\mu(\mu_*))$ must lie within the shaded region in Fig. 2.

It is now obvious that the ordinate y_*^{l} of the intersection of the lowest of the hyperbolas (4.1), (4.2) and the upper bound (4.8) is the lower bound on the Y-transformation of the shear modulus for a given value of ζ_1. Similarly, the ordinate y_*^{u} of the intersection of the highest of the hyperbolas (4.3), (4.4) and the lower bound (4.8) is the upper bound on the Y-transformation of the shear modulus for a given value of ζ_1. The values y_*^{l} and y_*^{u} depend on the elastic moduli of the phases and value of the parameter ζ_1, as it should be for the geometrical-parameter bounds on the shear modulus. However, y_*^{l} and y_*^{u} also depend on the conductivity constants σ_1 and σ_2 which we treated as a parameters of the problem. Thus, we can optimize the bounds over σ_1 and σ_2. This procedure is the opposite of the one used by Berryman and Milton (1988). They applied geometrical-parameter bounds on the conductivity and elastic moduli to obtain cross-property relations. We have used cross-property translation bounds and geometrical-parameter conductivity bounds to obtain geometrical-parameter bounds on the elastic moduli.

Let us now turn our attention to the lower bounds on the shear modulus. One must:

- find the points y_{1*} and y_{2*} of the intersection of the upper bound (2.21) with hyperbolas (4.1), and (4.2), respectively;
- find the lowest out of the points

$$y_*^{\mathrm{l}} = \min\{y_{1*}, y_{2*}\} \tag{4.12}$$

  as the bound on the value $y_\mu(\mu_*)$;
- optimize this value y_*^{l} over the parameters σ_1, σ_2, i.e.

$$y_\mu(\mu_*) \geqslant \max_{\sigma_1,\sigma_2} y_*^{\mathrm{l}}. \tag{4.13}$$

Let us now find the values y_{1*} and y_{2*}. Comparing the form of the upper bound (4.8)

$$y_\sigma(\sigma_*) = F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_2) \tag{4.14}$$

and the parameteric representation of the hyperbola (4.2)

$$y_\sigma(\sigma_*) = F(\sigma_1, \sigma_2, \gamma_1, \gamma_2, \sigma_2), \quad y_\mu(\mu_*) = F(y_1, y_2, \gamma_1, \gamma_2, \mu_2), \tag{4.15}$$

we can easily find that

$$y_{2*} = F(y_1, y_2, \zeta_1, \zeta_2, \mu_2). \tag{4.16}$$

In order to find y_{1*}, we find the value of the parameter γ_* as the solution of the equation

$$\sigma_*^{\mathrm{u}} = F(\sigma_1, \sigma_2, \gamma_*, 1-\gamma_*, \sigma_1) = \gamma_*\sigma_1 + (1-\gamma_*)\sigma_2 - \frac{\gamma_*(1-\gamma_*)(\sigma_1-\sigma_2)^2}{\gamma_*\sigma_2 + (1-\gamma_*)\sigma_1 + \sigma_1}, \tag{4.17}$$

where $\sigma_*^{\mathrm{u}} = F(\sigma_1, \sigma_2, \zeta_1, \zeta_2, \sigma_2)$ is the upper bound (4.8) for the given value of the parameter ζ_1. The solution yields

$$\gamma_* = \frac{2\sigma_1(\sigma_2 - \sigma_*^{\mathrm{u}})}{(\sigma_2-\sigma_1)(\sigma_1+\sigma_*^{\mathrm{u}})}. \tag{4.18}$$

Substituting the expression (4.9) for σ_*^{u} into (4.18) we find γ_* as a function of the parameter ζ_1:

$$\gamma_* = \frac{4\sigma_1\sigma_2\zeta_1}{(\sigma_1+\sigma_2)^2-\zeta_1(\sigma_1-\sigma_2)^2}. \tag{4.19}$$

Then we substitute this value into the expression for the ordinate of the point y_{1*}, i.e.

$$y_{1*} = \gamma_* y_1 + (1-\gamma_*)y_2 - \frac{\gamma_*(1-\gamma_*)(y_1-y_2)^2}{(1-\gamma_*)y_1+\gamma_* y_2+\mu_1}, \tag{4.20}$$

where γ_* is given by (4.19).

Now we recall that the values σ_1 and σ_2 can be treated as a parameters. Let us put $\sigma_1 = \sigma_2$. As follows from (4.19), (4.20) in this case

$$\gamma_* = \zeta_1, \quad 1-\gamma_* = \zeta_2, \tag{4.21}$$

and

$$y_{1*} = F(y_1, y_2, \zeta_1, \zeta_2, \mu_1) \tag{4.22}$$

As follows from (4.16) and (4.22) and from the monotonicity of the function $F(y_1, y_2, \zeta_1, \zeta_2, y)$ as a function of y,

$$y_* = \min\{F(y_1,y_2,\zeta_1,\zeta_2,\mu_1), F(y_1,y_2,\zeta_1,\zeta_2,\mu_2)\} = F(y_1,y_2,\zeta_1,\zeta_2,\mu_{\min}). \tag{4.23}$$

Comparing this result with the inequalities (2.36), (2.37) one can see that we improved upon the previously known bounds.

One can repeat the same procedure for the upper bound y_*^u and recover the upper bound (3.41), (3.42) that we proved in the previous section by using the translation method. Similarly, one can recover geometrical-parameter bulk modulus bounds by using cross-property conductivity-bulk modulus bounds found by Gibiansky and Torquato (1993, 1994) and Statement 2.

5. APPLICATIONS AND DISCUSSION

5.1. *Summary of the results*

Let us first summarize the new results for the geometrical-parameters that were obtained in the previous sections:

- We have presented geometrical-parameter bounds in a simple form using the Y-transformation. In this form the bounds do not depend explicitly on the volume fraction.
- We have improved the upper bound on the bulk modulus of a two-dimensional composite. This is now given by the inequality (3.37) or (3.38).
- We have improved the upper bound on the shear modulus of a two-dimensional composite that incorporates the geometrical parameters ζ_1, ζ_2, and η_1, η_2. This is now given by (3.50), (3.51).
- We have also improved both the upper and lower shear modulus bounds that incorporate only the parameters ζ_1, ζ_2. These are given by (3.42) and (4.23).

In this section we will study the attainability of the bounds and will apply these bounds to study the properties of particular composites.

5.2. *Attainability of the geometrical-parameter bounds*

We shall examine whether the geometrical-parameter bounds are attainable by certain structures. If the bounds can be shown to be attainable, then we know that they are optimal given that amount of structural information.

As was proved by Milton (1981b, 1984), the conductivity bounds (2.20) in two dimensions and the lower bound (2.21) in three dimensions are optimal since for any fixed value ζ_1, there exist structures that saturate the corresponding bound. These structures are the Hashin-type assemblages of doubly coated circles ($d = 2$) or spheres ($d = 3$). One can check that the same assemblages of doubly coated circles ($d = 2$) saturate also our new bulk modulus upper bound.

Five points along the upper bound (2.20) on the effective conductivity of three-dimensional isotropic composite are realized by five different structures (Milton 1981b, 1984). Two of them are trivial: the Hashin construction of coated spheres possesses the value $\zeta_1 = 1$ if phase 1 forms the coating and $\zeta_1 = 0$ if phase 1 is the included, core phase. Thus, these structures saturate the bounds (2.21). The other three structures are constructed by a two-step procedure. First, we prepare three different prototype composites: (i) the laminate composite of two phases, (ii) the Hashin assemblage of coated cylinders where phase 1 forms the coating, (iii) the same coated cylinder construction where phase 2 forms the coating. Next, we treat each of these prototype composites as a crystal and prepare three polycrystals from them by using Schulgasser's (1977) microstructures that maximize the effective conductivity of the polycrystal. The value of the parameter ζ_1 for these structures is given by

$$\zeta_1 = f_2, \quad \zeta_1 = 1-\tfrac{1}{4}f_1, \quad \zeta_1 = \tfrac{1}{4}f_2, \tag{5.1}$$

respectively (see Schulgasser, 1977, Milton, 1981b). The effective conductivity of these composites was found in the mentioned papers and in terms of Y-transformations are given by the formulas

$$y_\sigma(\sigma_*) = 2\sigma_1 f_2 + 2\sigma_2 f_1, \tag{5.2}$$

$$y_\sigma(\sigma_*) = 2\sigma_1(1-\tfrac{1}{4}f_1) + 2\sigma_2(\tfrac{1}{4}f_1), \tag{5.3}$$

$$y_\sigma(\sigma_*) = 2\sigma_1(\tfrac{1}{4}f_2) + 2\sigma_2(1-\tfrac{1}{4}f_2), \tag{5.4}$$

respectively. As can be easily seen, these structures saturate the bounds (2.21). The bulk moduli of these composites were calculated by Gibiansky and Milton (1993). Specifically, one has

$$y_\kappa(\kappa_*) = \tfrac{4}{3}\mu_1 f_2 + \tfrac{4}{3}\mu_2 f_1, \tag{5.5}$$

$$y_\kappa(\kappa_*) = \tfrac{4}{3}\mu_1(1-\tfrac{1}{4}f_1) + \tfrac{4}{3}\mu_2(\tfrac{1}{4}f_1), \tag{5.6}$$

$$y_\kappa(\kappa_*) = \tfrac{4}{3}\mu_1(\tfrac{1}{4}f_2) + \tfrac{4}{3}\mu_2(1-\tfrac{1}{4}f_2), \tag{5.7}$$

respectively. One can check that they saturate the upper bulk modulus bound (2.24). There also exists one microstructure (in addition to the Hashin coated circle or coated spheres assemblages) that saturate the lower bound (2.24) for the bulk modulus. This is a special polycrystal construction (that minimizes the effective bulk modulus of the polycrystal) made of laminates of the original phases, see Gibiansky and Milton (1993). The value $\zeta_1 = f_2$ for this structure can be calculated as in Milton (1981a), Schulgasser (1977) and is defined by

$$y_\kappa(\kappa_*) = \left[\frac{3f_2}{4\mu_1} + \frac{3f_1}{4\mu_2}\right]^{-1}, \quad \text{for } d = 3, \tag{5.8}$$

$$y_\kappa(\kappa_*) = \left[\frac{f_2}{\mu_1} + \frac{f_1}{\mu_2}\right]^{-1}, \quad \text{for } d = 2. \tag{5.9}$$

We note that the shear modulus bounds are only known to be attainable in the trivial cases when $\zeta_1 = \eta_1 = 1$ or $\zeta_1 = \eta_1 = 0$. Indeed, one can check that for these cases the structures that saturate Hashin–Shtrikman bounds on the shear modulus saturate the geometrical-parameter bounds as well.

5.3. *Comparing the old and new bounds for some particular composite structures*

Here we shall compare old bounds with our new bounds and inquire whether they can help to predict the effective properties of particular composites. We begin with composites having microstructures that correspond to the effective-medium theory geometries (Milton, 1984). The geometrical parameters for these structures are defined by the volume fraction, i.e. $\zeta_1 = \eta_1 = f_1$. It is known that effective bulk and shear moduli of such a composite can be found as a solution of the following system of equations:

$$f_1 \frac{\kappa_1 - \kappa_e}{\kappa_1 + \mu_e} + f_2 \frac{\kappa_2 - \kappa_e}{\kappa_2 + \mu_e} = 0, \tag{5.10}$$

$$f_1 \frac{\mu_1 - \mu_e}{\mu_1 + \kappa_e\mu_e/(\kappa_e + 2\mu_e)} + f_2 \frac{\mu_2 - \mu_e}{\mu_2 + \kappa_e\mu_e/(\kappa_e + 2\mu_e)} = 0. \tag{5.11}$$

As was noted by Berryman (1982), the Y-transformation has an application to the effective-medium theory. Namely, he noted that for the conductivity problem the effective-medium-theory geometry is an "eigenvalue" of the Y-transformation, i.e.

$$y_\sigma(\sigma_*) = (d-1)\sigma_*. \tag{5.12}$$

This can be generalized to elasticity as well, i.e. the system of (5.10), (5.11) can be rewritten in the form

$$y(\kappa_e) = \mu_e, \quad y(\mu_e) = \kappa_e\mu_e/(\kappa_e + 2\mu_e), \quad (d = 2). \tag{5.13}$$

The expressions on the right-hand sides of (5.13) have the form of the Y-transformations of the effective moduli of the Hashin assemblages of coated spheres (circles).

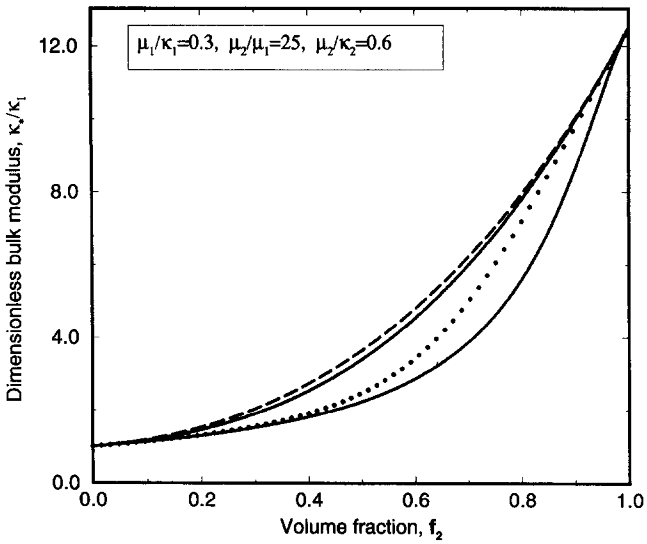

Fig. 3. Geometrical-parameter bounds on the effective bulk modulus of composites with $\zeta_1 = \eta_1 = f_1$, i.e. corresponding to the effective medium theory geometries. The dotted line is the exact result, solid lines are our new bounds and the dashed line is the Silnutzer upper bound (Silnutzer lower bound coincides with our bound).

Figure 3 depicts the old and new bulk modulus bounds, (2.24) and (3.37), respectively, as a function of the volume fraction f_2 for the following values of the parameters

$$\mu_1/\kappa_1 = 0.3, \quad \mu_2/\mu_1 = 25, \quad \mu_2/\kappa_2 = 0.6. \tag{5.14}$$

We did not improve the lower bound. The new upper bound is tighter than the old one. In fact, it is optimal as we mentioned in the previous section.

Table 1 illustrates the old and new shear moduli bounds. Column 1 gives the volume fraction f_2 of phase 2, columns 2 $[\mu_u^{old}(\zeta_1)]$ and 9 $[\mu_l^{old}(\zeta_1)]$ correspond to the Kublanov–

Table 1. *Comparison of the bounds on the dimensionless effective shear modulus μ_*/μ_1 for a composite with $\zeta_1 = \eta_1 = f_1$ which correspond, for example, to the effective-medium theory geometries*

f_2	$\mu_u^{old}(\zeta_1)$	$\mu_u^{new}(\zeta_1)$	$\mu_u^{old}(\zeta_1,\eta_1)$	$\mu_u^{new}(\zeta_1,\eta_1)$	μ_{eff}	$\mu_l^{old}(\zeta_1,\eta_1)$	$\mu_l^{new}(\zeta_1)$	$\mu_l^{old}(\zeta_1)$
0.10	1.386	1.318	1.271	1.270	1.177	1.174	1.171	1.170
0.30	3.057	2.626	2.624	2.590	1.785	1.722	1.666	1.661
0.50	6.063	5.299	5.433	5.297	3.291	2.848	2.546	2.531
0.60	8.209	7.403	7.589	7.403	4.973	3.922	3.282	3.261
0.70	10.930	10.211	10.408	10.211	7.929	5.760	4.423	4.396
0.80	14.411	13.915	14.069	13.915	12.465	9.178	6.400	6.369
0.85	16.521	16.175	16.287	16.175	15.236	11.956	8.039	8.009
0.90	18.941	18.751	18.815	18.751	18.286	15.761	10.587	10.562
0.95	21.739	21.680	21.700	21.680	21.553	20.509	15.072	15.058

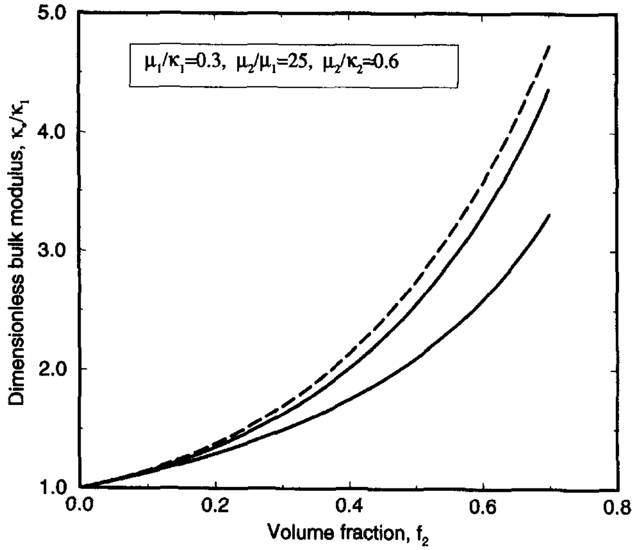

Fig. 4. The same as Fig. 3 but for the random array of disks of Torquato and Lado (1988, 1992).

Milton bounds in the form (2.36) that exclude the η-parameter. Subscripts u and l denote the upper and lower bound, respectively. Columns 4 $[\mu_u^{old}(\zeta_1,\eta_1)]$ and 7 $[\mu_l^{old}(\zeta_1,\eta_1)]$ give the Kublanov–Milton bounds (2.32). Columns 3 $[\mu_u^{new}(\zeta_1)]$ and 8 $[\mu_l^{new}(\zeta_1)]$ give our new bounds (3.42) and (4.23), respectively. Column 5 gives the new bound (3.50) $[\mu_u^{new}(\zeta_1,\eta_1)]$ that incorporates the parameter η_1. Finally, column 6 $[\mu_{eff}]$ gives the exact effective shear modulus of the effective-medium theory geometry.

Our shear modulus bounds are seen to provide improvement over known bounds. Interestingly, for large volume fraction f_2, the optimal upper bound does not depend on the parameter η_1. Indeed, the new upper bounds in the form (3.42) (which does not incorporate η_1) and in the form (3.50) (which incorporates η_1) coincide for $f_2 \geqslant 0.6$ for the chosen values (5.14) of the material parameters. Both of them provide improvement over the bound (2.32).

Figure 4 and Table 2 present similar bounds for the random array of hard disks

Table 2. *Comparison of the bounds on the dimensionless effective shear modulus* μ_*/μ_1 *for the random array of disks* (*Torquato and Lado* (1988, 1992)).

f_2	$\mu_u^{old}(\zeta_1)$	$\mu_u^{new}(\zeta_1)$	$\mu_u^{old}(\zeta_1,\eta_1)$	$\mu_u^{new}(\zeta_1,\eta_1)$	$\mu_l^{old}(\zeta_1,\eta_1)$	$\mu_l^{new}(\zeta_1)$	$\mu_l^{old}(\zeta_1)$
0.10	1.302	1.276	1.225	1.225	1.171	1.169	1.169
0.20	1.764	1.658	1.602	1.599	1.390	1.378	1.378
0.30	2.422	2.182	2.164	2.139	1.677	1.645	1.643
0.40	3.329	2.900	2.970	2.882	2.063	1.994	1.990
0.50	4.564	3.893	4.107	3.892	2.603	2.467	2.461
0.60	6.250	5.292	5.707	5.292	3.396	3.146	3.135
0.70	8.587	7.323	7.989	7.323	4.643	4.196	4.178

(aligned cylinders in three dimensions) using the geometrical parameter values given for such structures by Torquato and Lado (1988, 1992), i.e.

$$\zeta_2 = 0.33333f_2 - 0.05707f_2^2, \quad \eta_2 = 0.69148f_2 + 0.04280f_2^2. \tag{5.15}$$

The values of the material parameters are also chosen according to (5.14). In Table 2, we use the same notation as in Table 1. Again, we see that our bounds provide improvement over previously known ones.

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REFERENCES

Beran, M. (1965) Use of the variational approach to determine bounds for the effective permittivity in random media. *Nuovo Cimento* **38,** 771–782.

Beran, M. J. and Molyneux, J. (1966) Use of classical variational principles to determine bounds for the effective bulk modulus in heterogeneous media. *Q. Appl. Math.* **24,** 107–118.

Berryman, J. G. (1982) Effective medium theory for elastic composites. *Elastic Wave Scattering and Propagation* (ed. V. K. Varadan and V. V. Varadan), pp. 111–129. Ann Arbor, Michigan.

Berryman, J. G. and Blair, S. C. (1986) Use of digital image analysis to estimate fluid permeability of porous materials: applications of two-point correlation functions. *J. Appl. Phys.* **60,** 1930–1938.

Berryman, J. G. and Milton, G. W. (1988) Microgeometry of random composites and porous media. *J. Phys. D: Appl. Phys.* **21,** 87–94.

Cherkaev, A. V. and Gibiansky, L. V. (1992) The exact coupled bounds for effective tensors of electrical and magnetic properties of two-component two-dimensional composites. *Proc. Roy. Soc. Edinburgh* **122A,** 93–125.

Cherkaev, A. V. and Gibiansky, L. V. (1993) Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite. *J. Mech. Phys. Solids* **41**(5), 937–980.

Davis, L. C. (1991) Third-order bounds on the elastic moduli of metal-matrix composites. *Metall. Trans. A* **22A,** 3065–3068.

Davis, L. C., Chen, J. and Thorpe, M. F. (1992) Predicting the elastic properties of composite materials. *Proc. Amer. Soc. Comp., 7th Tech. Conf.*, Un. Park, PA.

Francfort, G. A. and Murat, F. (1986) Homogenization and optimal bounds in linear elasticity. *Arch. Rat. Mech. Anal.* **94,** 307–334.

Eischen, J. W. and Torquato, S. (1993) Determining elastic behavior of composites by the boundary element method. *J. Appl. Phys.* **74**(1), 159–170.

Gibiansky, L. V. and Milton, G. W. (1993) On the effective viscoelastic moduli of two-phase media: I. Rigorous bounds on the complex bulk modulus. *Proc. Roy. Soc. London A* **440,** 163–188.

Gibiansky, L. V. and Torquato, S. (1993) Link between the conductivity and elastic moduli of composites. *Phys. Rev. Lett.* **71,** 2927–2930.

Gibiansky, L. V. and Torquato, S. (1994) Rigorous link between the conductivity and elastic moduli of fiber-reinforced composite materials. *Trans. Roy. Soc. London A* (in press).

Hashin, Z. and Shtrikman, S. (1962) A variational approach to the theory of the effective magnetic permeability of multiphase materials. *J. Appl. Phys.* **35,** 3125–3131.

Hashin, Z. and Shtrikman, S. (1963) A variational approach to the theory of the elastic behaviour of multiphase materials. *J. Mech. Phys. Solids* **11,** 127–140.

Hashin, Z. (1965) On elastic behaviour of fibre reinforced materials of arbitrary transverse phase geometry. *J. Mech. Phys. Solid.* **13,** 119–134.

Helsing, J. (1993) Improved bounds on the conductivity of composites by translation in a variational principle. *J. Appl. Phys.* **74,** 5061–5063.

Helsing, J. (1994a) Bounds on the shear modulus of composites by interface integral methods. *J. Mech. Phys. Solids* **42,** 1123–1138.

Helsing, J. (1994b) Third-order bounds on the conductivity of a random stacking of cubes. *J. Math Phys.* **35,** 1688–1692.

Hill, R. (1964) Theory of mechanical properties of fiber-strengthened materials—I. Elastic behavior. *J. Mech. Phys. Solids* **12,** 199–212.

Kublanov, L. and Milton, G. W. (1991) Rigorous estimates for the elastic moduli for a two-dimensional two-phase material, unpublished manuscript.

Lurie, K. A. and Cherkaev, A. V. (1984) Exact estimates of conductivity of composites formed by two isotropically conducting media taken in prescribed proportion. *Proc. Roy. Soc. Edinburgh* **99 A,** 71–87.

Lurie, K. A. and Cherkaev, A. V. (1985) The problem of formation of an optimal multicomponent composite. *J. Opt. Th. Appl.* **46,** 571–589.

Lurie, K. A. and Cherkaev, A. V. (1986) Exact estimates of the conductivity of a binary mixture of isotropic materials. *Proc. Roy. Soc. Edinburgh,* **104 A,** 21–38.

McCoy, J. J. (1970) On the displacement field in an elastic medium with random variations of material properties. *Recent Advances in Engineering Sciences* 5, Pt. 2, pp. 235–254. Gordon and Breach, New York.

Milton, G. W. (1981a) Bounds on the electromagnetic, elastic, and other properties of two-component composites. *Phys. Rev. Lett.* **46,** 542–545.

Milton, G. W. (1981b) Bounds on the transport and optical properties of a two-component composite material. *J. Appl. Phys.* **52,** 5294–5304.

Milton, G. W. (1982) Bounds on the elastic and transport properties of two-component composites. *J. Mech. Phys. Solids* **30,** 177–191.

Milton, G. W. (1984) Correlation of the electromagnetic and elastic properties of composites and microgeometries corresponding with effective medium approximations. *Physics and Chemistry of Porous Media* (ed. D. L. Johnson and P. N. Sen). American Institute of Physics.

Milton, G. W. (1986) Modeling the properties of composites by laminates. *Homogenization and Effective Moduli of Materials and Media* (ed. J. L. Eriksen *et al.*), pp. 150–174. Springer-Verlag, New York.

Milton, G. W. (1990) On characterizing the set of possible effective tensors of composites: the variational method and the translation method. *Comm. Pure Appl. Math.* **43,** 63–125.

Milton, G. W. (1991) The field equation recursion method. *Composite Media and Homogenization Theory* (ed. G. Dal Maso and G. Dal Antonio), Vol. 5, pp. 223–245. Birkhauser, Boston.

Milton, G. W. (1993) private communication.

Milton, G. W. and Phan-thien, N. (1982) New bounds on the effective moduli of two-component materials. *Proc. R. Soc. London A* **380,** 305–331.

Murat, F. and Tartar, L. (1985) Calcul des variations et homogeneisation. *Les Methodes de l'Homogeneisation: Theorie et Applications en Physique*, pp. 319–370. Coll. de la Dir. des Etudes et Recherches de Electr. de France, Eyrolles, Paris.

Norris, A. N. (1985) A differential scheme for the effective moduli of composites. *Mech. Mater.* **4,** 1–16.

Prager, S. (1963) Diffusion and viscous flow in concentrated suspensions. *Physica* **29,** 129–139.

Quintanilla, J. and Torquato, S. (1995) New bounds on the elastic moduli of suspensions of spheres. *J. Appl. Phys.* **77**(9), 4361–4372.

Silnutzer, N. (1972) Effective constants of statistically homogeneous materials. Ph.D. Thesis, University of Pennsylvania, Philadelphia.

Schulgasser, K. (1976a) On the conductivity of fiber-reinforced materials. *J. Math. Phys.* **17,** 382–387.

Schulgasser, K. (1976b) Relationship between single-crystal and polycrystal electrical conductivity. *J. Appl. Phys.* **47,** 1880–1886.

Schulgasser, K. (1977) Bounds on the conductivity of statically isotropic polycrystals. *J. Phys. C* **10,** 407–417.

Tartar, L. (1985) Estimations fines des coefficients homogeneises. *Ennio de Giorgi Colloquium* (ed. P. Kree), pp. 168–187. Pitman Research Notes in Math, 125.

Torquato, S. (1980) Microscopic approach to transport in two-phase random media. Ph.D. Dissertation, State University of New York, Stony Brook, New York.

Torquato, S. (1985) Electrical conductivity of two-phase disordered composite media. *J. Appl. Phys.* **58,** 3790–3797.

Torquato, S. (1991) Random heterogeneous media: microstructure and improved bounds on effective properties. *Appl. Mech. Rev.* **44**(2), 37–76.

Torquato, S. and Lado, F. (1988) Bounds on the conductivity of a random array of cylinders. *Proc. Roy. Soc. London A* **417,** 59–80.

Torquato, S. and Lado, F. (1992) Improved bounds on the effective elastic moduli of random arrays of cylinders. *J. Appl. Mech.* **59,** 1–6.

Torquato, S. and Stell, G. (1982) Microstructure of two-phase random media. I. The n-point probability functions. *J. Chem. Phys.* **77,** 2071–2077.

Walpole, L. J. (1966) On bounds for the overall elastic moduli of inhomogeneous systems—I. *J. Mech. Phys. Solids* **14,** 151–162.